DESCRIPTION OF THE RESEARCH

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1. Previous research

1.1. **Equivariant quantization.** My research is linked to quantization, a concept which comes from quantum physics and which consists in associating with a classical observable a quantum observable. A quantization allows one to establish a correspondence between the classical formalism and the quantum formalism of Mechanics. From a mathematical and geometrical point of view, a classical observable can be identified with a polynomial function on the cotangent bundle T^*M of a manifold M (a symbol) whereas a quantum observable can be assimilated with a differential operator defined on M.

There is no quantization that commutes with the action of all the diffeomorphisms of M, but when a Lie group G acts on M, we define a G-equivariant quantization as a linear bijection preserving the principal symbol and exchanging the action of G between a space of symbols and the corresponding space of differential operators. One takes G small enough to have the existence of the quantization but large enough to have its uniqueness.

When M is the projective space and G is the projective group, this leads to the notion of projectively equivariant quantization, whereas when M is equal to $S^p \times S^q$ and G is equal to $SO(p+1, q+1)$, this leads to the notion of conformally equivariant quantization. From an infinitesimal point of view, the projectively and conformally equivariant quantizations correspond to quantizations defined on the space $\mathbb{R}^m$ that commute with the Lie derivative in the direction of vector fields belonging to maximal Lie subalgebras of the Lie algebra of vector fields on $\mathbb{R}^m$. The existence of projective and conformal quantizations over $\mathbb{R}^m$ has been proved for many spaces of differential operators (see e.g. [22] or [14]). Other invariance types have been studied too; in all these situations, the existence of quantizations on $\mathbb{R}^m$ has been proved ([4]).

The concept of equivariant quantization on $\mathbb{R}^m$ has a counterpart on an arbitrary manifold ([21]). In the projective situation, it consists of the quest for a quantization depending on a connection but only on its projective class, the quantization being natural in all of its arguments. Such a quantization is called a natural and projectively invariant quantization. In the conformal situation, the quantization depends on a pseudo-Riemannian metric, but only on its conformal class, the quantization being natural in all of its arguments. Such a quantization is called a natural and conformally invariant quantization. I solved completely the problem of the natural and projectively invariant quantization and the problem of the natural and conformally invariant quantization ([26, 27, 40, 41, 29, 28, 42]) thanks to the theory of parabolic geometries exposed in [9] and in [10].

After these works, I began to investigate the quantization of singular spaces by proving in [39] the existence of equivariant quantizations on orbifolds using a desingularization technique and the existence of foliated quantizations established in [38].

1.2. **Supergeometry.** Another direction of my research is the quest for invariant quantizations on supermanifolds. In [30] (resp. in [23]), we proved the existence of an $\mathfrak{sl}(p+1|q)$ (resp. $\mathfrak{osp}(p+1, q+1|2r)$)-equivariant quantization on $\mathbb{R}^{p|q}$ (resp. $\mathbb{R}^{p+q|2r}$), generalizing in this way the projectively (resp. conformally) equivariant quantization over $\mathbb{R}^m$. In [35], we show that there exists a unique $\mathfrak{spo}(2|2)$-equivariant quantization on the supercircle $S^{1|2}$, where $\mathfrak{spo}(2|2)$ is a Lie algebra consisting of contact projective vector fields. In [24], we prove the existence of a natural and projectively invariant quantization on supermanifolds for differential operators acting between densities, adapting a method used by M. Bordemann ([5]).

In [25], we study the notion of geodesic in supergeometry: we give a definition of a geodesic corresponding to a connection on a supermanifold and we prove notably that two connections are projectively equivalent in the algebraic sense of [24] if and only if they have the same geodesics up to parametrization, proving thus that the Weyl characterization holds too in the super setting.

1.3. **Conformal symmetries of the conformal Laplacian.** The natural and conformally invariant quantization exposed previously allowed me to study in [32] with J.-P. Michel and J. Silhan the symmetries and the conformal symmetries of the conformal Laplacian Δ_Y on a pseudo-Riemannian manifold (M, g). On an n-dimensional pseudo-Riemannian manifold (M, g), the conformal Laplacian Δ_Y is defined in this way:

$$\Delta_Y = \Delta - \frac{n-2}{4(n-1)}\mathrm{Sc},$$

where Δ denotes the Laplace-Beltrami operator and Sc denotes the scalar curvature associated with g.

A symmetry of Δ_Y is a differential operator that commutes with Δ_Y. A conformal symmetry of Δ_Y is a generalization of this concept: it is a differential operator D_1 such that there exists a differential operator D_2 giving rise to the relation $\Delta_Y \circ D_1 = D_2 \circ \Delta_Y$. The existence of (conformal) symmetries of differential operators is linked to the existence of R-separating coordinates systems for some partial differential equations (see e.g. [19] or [6]). The concept of R-separation allows one to reduce the resolution of these partial differential equations to the resolution of ordinary differential equations.

The conformal symmetries of the conformal Laplacian were completely known on conformally flat manifolds thanks to works by M. Eastwood and J.-P. Michel (see [15] and [33]). On an Einstein pseudo-Riemannian manifold endowed with a Killing tensor K, B. Carter proved in [11] the existence of a second-order symmetry of Δ_Y that has K as principal symbol. But until now, the (conformal) symmetries of Δ_Y were unknown on an arbitrary pseudo-Riemannian manifold, even at the second order.

In [32], we describe thanks to the natural and conformally invariant quantization all the second-order (conformal) symmetries of Δ_Y on an arbitrary pseudo-Riemannian manifold (M, g). The principal symbol of such a (conformal) symmetry has to be a symmetric (conformal) Killing 2-tensor that satisfies some additional condition. We then determine whether this condition is verified on pseudo-Riemannian manifolds endowed with (conformal) Killing tensors. As explained in the scientific project, we expect that the conformal symmetries of Δ_Y allow one to characterize the existence of R-separating coordinate systems for the Schrödinger equation at zero energy $\Delta_Y\psi = 0$, exactly in the same way as the conformal symmetries of the Laplace-Beltrami operator Δ allow one to characterize the existence of R-separating coordinate systems for the Laplace equation $\Delta_Y\psi = 0$ (see [19]).

2. Research project

2.1. **Equivariant quantizations in supergeometry.** In a first step, I would like to continue to explore the equivariant quantization in supergeometry. This project is then in the prolongation of the works [30] and [23] in the flat setting and of the work [24] in the curved setting. More precisely, I would like to describe in a first step the different maximal Lie subalgebras in the Lie superalgebra of supervector fields on the superspace $\mathbb{R}^{p|q}$ and to study the corresponding equivariant quantizations, adapting in this way to the context of the supergeometry the work [4] in which F. Boniver and P. Mathonet solved the problem of the IFFT-equivariant quantizations over $\mathbb{R}^m$, where the IFFT Lie algebras are graded Lie algebras classified by Kobayashi and Nagano in [20]. In a second step, I would like to define the notion of parabolic geometry in supergeometry and to build the Cartan fiber bundles and connections associated with these geometries. These objects could in particular allow one to extend the work [24] to the conformal setting.

2.2. **Equivariant quantizations for parabolic geometries.** In [35], I proved with N. Mellouli and A. Nibirantiza the existence and the uniqueness of an $\mathfrak{spo}(2|2)$-equivariant quantization on the supercircle $S^{1|2}$ for differential operators acting between λ- and μ-densities when $\delta = \mu - \lambda$ is different from some resonant values. After this work, it remained to generalize this result to a contact supermanifold of arbitrary superdimension. This task is actually almost completed; I began recently to write an article about this subject with J.-P. Michel and A. Nibirantiza ([34]). It could be very interesting to generalize the contact projectively equivariant quantization studied in [13], [18], [31] and [35] to an arbitrary contact manifold using the concept of contact projective structure described e.g. by D. Fox in [17]. This concept is certainly the counterpart in the contact projective setting of the concept of projective (resp. conformal) structure in the projective (resp. conformal) setting.

The contact projective geometry is a particular case of parabolic geometry. The parabolic geometries are Cartan geometries of type (G, P) for semisimple Lie groups G and parabolic subgroups P; they are described e.g. in [8]. The Lie algebra $\mathfrak{g}$ of the Lie group G carries a $|k|$-grading; if $k = 1$, $\mathfrak{g}$ is among the IFFT Lie algebras classified in [20]. In this situation, the problem of equivariant quantization is completely solved (see [7] or [29]). My next project is to extend the concept of equivariant quantization to parabolic geometries for which the semisimple Lie algebra $\mathfrak{g}$ carries a $|k|$-grading with $k \geq 2$ (contact projective geometry and CR-geometry are examples of such geometries). The filtration on the space of differential operators that is well adapted to this problem (i.e. the filtration that implies the uniqueness of the equivariant quantization) is certainly the filtration described in [36] thanks to the filtration of the tangent bundle associated with the parabolic geometry.

2.3. **R-separation of variables.** As explained in the review on my previous research, I describe in [32] with J.-P. Michel and J. Silhan the structure of second-order (conformal) symmetries of the conformal Laplacian Δ_Y on a pseudo-Riemannian manifold (M, g). A necessary and sufficient condition to have the existence of a second-order (conformal) symmetry of Δ_Y is the existence of a (conformal) Killing tensor of degree 2 satisfying an additional property.

The next step in the study of the (conformal) symmetries of Δ_Y is the study of the (conformal) symmetries of Δ_Y at an arbitrary order.

In [19], the authors give a necessary and sufficient condition for the R-separation of the Laplace equation $\Delta\psi = 0$, where Δ denotes the Laplace-Beltrami operator, in terms of existence of second-order conformal symmetries of Δ. We can thus expect to find a necessary and sufficient condition for the R-separation in an orthogonal coordinates system of the Schrödinger equation at zero energy $\Delta_Y\psi = 0$ in terms of existence of second-order conformal symmetries of Δ_Y.

From there, many projects can be imagined. We can first try to describe all the R-separating coordinates systems for the equation $\Delta_Y\psi = 0$ on a pseudo-Riemannian manifold. A necessary condition for the existence of such a separating coordinate system is the fact that the metric g is conformal Stäckel (see e.g. [12]). A first step in the description of the R-separating coordinates sytems is then the classification of all pseudo-Riemannian manifolds endowed with conformal Stäckel metrics. We could try in a second step to determine if the coordinates systems corresponding to these metrics (i.e. the coordinates systems allowing one the separation of the Hamilton-Jacobi equation) lead to the R-separation of $\Delta_Y\psi = 0$ by analyzing the condition linked to the existence of conformal symmetries of Δ_Y for these metrics. We could even try to determine a necessary and sufficient condition for the R-separation of the equation $\Delta_Y\psi = 0$ only in terms of the metric g. Such a condition would be analogous to the Robertson's condition presented in [16] which characterizes the existence of a R-separating coordinates system for the equation $\Delta\psi = E\psi$ only in terms of the Ricci tensor associated with the metric g.

The separation of variables is a very fashionable field of study: in [43], the authors describe the global geometry of the set of separating coordinates systems for the Hamilton-Jacobi equation on the sphere S^n endowed with the canonical metric. It could be very interesting to extend this result to an arbitrary pseudo-Riemannian manifold and to adapt it in order to study the geometrical structure of the set of R-separating coordinates systems for the Schrödinger equation $\Delta_Y\psi = E\psi$.

2.4. **Integral formulae for the equivariant quantizations.** Another direction of research is the quest of integral formulae for the equivariant quantizations. Such formulae were often used to express quantizations in the framework of deformation quantization (see e.g. [1, 2, 3]). The advantages of an integral formula are the following: first, such a formula has a compact form whereas we do not have in general explicit and simple formulae for the equivariant quantizations. Secondly, the invariance property of the equivariant quantizations would certainly appear much more clearly on an integral formula. Eventually, such a formula allows one to extend the quantization to symbols of pseudo-differential operators. A first step in the quest of integral formulae for the equivariant quantizations was taken in the work [37] by M. Pevzner and A. Unterberger. This article is devoted to harmonic analysis and pseudo-differential analysis on the homogeneous space $X_n = SL(n+1,\mathbb{R})/GL(n,\mathbb{R})$. In this paper, the authors build an $SL(n+1,\mathbb{R})$-equivariant quantization of the space X_n with values in a space of pseudo-differential operators acting between densities defined on the n-dimensional projective space. Despite of the fact that the space of symbols considered in [37] is not exactly the same as the space of symbols considered in the framework of equivariant quantization, the method used in [37] can be certainly adapted to build integral formulae for the $\mathfrak{sl}(n+1,\mathbb{R})$-equivariant quantization introduced by C. Duval, P. Lecomte and V. Ovsienko. In a second step, we could even try to adapt the method for other types of invariance or for the curved setting.

References

[1] P. Bieliavsky, S. Detournay, and Ph. Spindel. The deformation quantizations of the hyperbolic plane. *Comm. Math. Phys.*, 289(2):529–559, 2009.

[2] Pierre Bieliavsky, Axel de Goursac, and Gijs Tuynman. Deformation quantization for Heisenberg supergroup. *J. Funct. Anal.*, 263(3):549–603, 2012.

[3] Pierre Bieliavsky, Razvan Gurau, and Vincent Rivasseau. Noncommutative field theory on rank one symmetric spaces. *J. Noncommut. Geom.*, 3(1):99–123, 2009.

[4] F. Boniver and P. Mathonet. IFFT-equivariant quantizations. *J. Geom. Phys.*, 56(4):712–730, 2006.

[5] M. Bordemann. Sur l'existence d'une prescription d'ordre naturelle projectivement invariante. *arXiv:math/0208171 [math.DG]*, 2002.

[6] C. P. Boyer, E. G. Kalnins, and W. Miller, Jr. Symmetry and separation of variables for the Helmholtz and Laplace equations. *Nagoya Math. J.*, 60:35–80, 1976.

[7] A. Čap and J. Šilhan. Equivariant quantizations for AHS-structures. *Adv. Math.*, 224(4):1717 – 1734, 2010.

[8] A. Čap and J. Slovák. *Parabolic Geometries I: Background and General Theory*, volume 154 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2009.

[9] A. Čap, J. Slovák, and V. Souček. Invariant operators on manifolds with almost Hermitian symmetric structures. I. Invariant differentiation. *Acta Math. Univ. Comenian. (N.S.)*, 66(1):33–69, 1997.

[10] A. Čap, J. Slovák, and V. Souček. Invariant operators on manifolds with almost Hermitian symmetric structures. II. Normal Cartan connections. *Acta Math. Univ. Comenian. (N.S.)*, 66(2):203–220, 1997.

[11] B. Carter. Killing tensor quantum numbers and conserved currents in curved space. *Phys. Rev. D (3)*, 16(12):3395–3414, 1977.

[12] C. Chanu and G. Rastelli. Fixed energy R-separation for Schrödinger equation. *Int. J. Geom. Methods Mod. Phys.*, 3(3):489–508, 2006.

[13] Charles H. Conley and Valentin Ovsienko. Linear differential operators on contact manifolds. *arXiv:1205.6562*, 2012.

[14] C. Duval, P. B. A. Lecomte, and V. Yu. Ovsienko. Conformally equivariant quantization: existence and uniqueness. *Ann. Inst. Fourier (Grenoble)*, 49(6):1999–2029, 1999.

[15] M. G. Eastwood. Higher symmetries of the Laplacian. *Ann. of Math. (2)*, 161(3):1645–1665, 2005.

[16] Luther Pfahler Eisenhart. Separable systems of Stackel. *Ann. of Math. (2)*, 35(2):284–305, 1934.

[17] Daniel J. F. Fox. Contact projective structures. *Indiana Univ. Math. J.*, 54(6):1547–1598, 2005.

[18] H. Gargoubi, N. Mellouli, and V. Yu. Ovsienko. Differential operators on supercircle: conformally equivariant quantization and symbol calculus. *Lett. Math. Phys.*, 79(1):51–65, 2007.

[19] E. G. Kalnins and W. Miller, Jr. Intrinsic characterisation of orthogonal R-separation for Laplace equations. *J. Phys. A*, 15(9):2699–2709, 1982.

[20] S. Kobayashi and T. Nagano. On filtered Lie algebras and geometric structures. I. *J. Math. Mech.*, 13:875–907, 1964.

[21] P. B. A. Lecomte. Towards projectively equivariant quantization. *Progr. Theoret. Phys. Suppl.*, (144):125–132, 2001. Noncommutative geometry and string theory (Yokohama, 2001).

[22] P. B. A. Lecomte and V. Yu. Ovsienko. Projectively equivariant symbol calculus. *Lett. Math. Phys.*, 49(3):173–196, 1999.

[23] T. Leuther, P. Mathonet, and F. Radoux. On $\mathfrak{osp}(p+1, q+1|2r)$-equivariant quantizations. *J. Geom. Phys.*, 62(1):87 – 99, 2012.

[24] Thomas Leuther and Fabian Radoux. Natural and projectively invariant quantizations on supermanifolds. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 7:12, 2011.

[25] Thomas Leuther, Fabian Radoux, and Gijs M. Tuynman. Geodesics on a supermanifold and projective equivalence of superconnections. *J. Geom. Phys.*, 67:81–96, 2013.

[26] P. Mathonet and F. Radoux. Natural and projectively equivariant quantizations by means of Cartan connections. *Lett. Math. Phys.*, 72(3):183–196, 2005.

[27] P. Mathonet and F. Radoux. Cartan connections and natural and projectively equivariant quantizations. *J. Lond. Math. Soc. (2)*, 76(1):87–104, 2007.

[28] P. Mathonet and F. Radoux. On natural and conformally equivariant quantizations. *J. Lond. Math. Soc., II. Ser.*, 80(1):256–272, 2009.

[29] P. Mathonet and F. Radoux. Existence of natural and conformally invariant quantizations of arbitrary symbols. *J. Nonlinear Math. Phys.*, 17:539–556, 2010.

[30] Pierre Mathonet and Fabian Radoux. Projectively equivariant quantizations over the superspace $\mathbb{R}^{p|q}$. *Lett. Math. Phys.*, 98(3):311–331, 2011.

[31] N. Mellouli. Second-order conformally equivariant quantization in dimension 1|2. *SIGMA*, 5:111, 2009.

[32] J.-P. Michel, F. Radoux, and J. Šilhan. Second order symmetries of the conformal Laplacian. *submitted in SIGMA, arXiv:1308.1046*, 2013.

[33] J.-Ph. Michel. Higher symmetries of Laplacian via quantization. *arXiv:1107.5840*, 2011.

[34] Nibirantiza A. Radoux F. Michel, J.-P. On the contact projectively equivariant quantization on a contact supermanifold. *In preparation*, 2013.

[35] A. Nibirantiza N. Mellouli and F. Radoux. $\mathfrak{spo}(2|2)$-equivariant quantizations on the supercircle $s^{1|2}$. *SIGMA*, 9:17, 2013.

[36] Katharina Neusser. Universal prolongation of linear partial differential equations on filtered manifolds. *Archivum Mathematicum*, 45(4):289–300, 2009.

[37] Michael Pevzner and André Unterberger. Projective pseudodifferential analysis and harmonic analysis. *J. Funct. Anal.*, 242(2):442–485, 2007.

[38] N. Poncin, F. Radoux, and R. Wolak. A first approximation for quantization of singular spaces. *J. Geom. Phys.*, 59(4):503–518, 2009.

[39] N. Poncin, F. Radoux, and R. Wolak. Equivariant quantization of orbifolds. *J. Geom. Phys.*, 60(9):1103–1111, 2010.

[40] F. Radoux. Explicit formula for the natural and projectively equivariant quantization. *Lett. Math. Phys.*, 78(2):173–188, 2006.

[41] F. Radoux. Non-uniqueness of the natural and projectively equivariant quantization. *J. Geom. Phys.*, 58(2):253–258, 2008.

[42] F. Radoux. An explicit formula for the natural and conformally invariant quantization. *Lett. Math. Phys.*, 89(3):249–263, 2009.

[43] K. Schöbel and A. P. Veselov. Separation coordinates, moduli spaces and stasheff polytopes. *arXiv:1307.6132*, 2013.