# Conformally equivariant quantization of supercotangent bundles

Jean-Philippe Michel

Institut Camille Jordan & Université Claude Bernard Lyon 1

# Quantization

| | classical | quantum |
|---|---|---|
| Phase space | $(\mathcal{M}, \omega)$ | $\mathcal{H}$ |
| Observables | $A \subset \mathcal{C}^\infty(\mathcal{M})$ | $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ |
| Symmetry | $\mathfrak{g} \subset \mathrm{ham}(\mathcal{M}, \omega)$ | $\mathfrak{g} \subset \mathrm{u}(\mathcal{H})$. |

## Example

1. Geometric quantization : $[\mathcal{Q}_G(f), \mathcal{Q}_G(g)] = \frac{\hbar}{i}\mathcal{Q}_G(\{f, g\})$, few quantizable observables.
2. Deformation quantization : $[Q(f), Q(g)] = \frac{\hbar}{i}Q(\{f, g\}) + O(\hbar^2)$, always exists, far to be unique.

# Equivariant quantization [Duval, Lecomte, Ovsienko '99]

Idea : fix $\mathcal{Q}$ by symmetries of the config. space $M$

| classical | quantum |
|---|---|
| $T^*M$ with $G$-flat $M$ | $\mathrm{L}^2(M)$ |
| Symbols $\mathcal{S}(M)$ | Diff. Op. $\mathcal{D}(M)$ |
| Hamiltonian action of $\mathfrak{g}$ | adjoint action of $\mathfrak{g}$ |

$\mathfrak{g}$-equivariant quantization $=$ ($\mathfrak{g}$-equivariant symbol map)$^{-1}$
$\mathcal{Q}(J_X) = L_X$ and $[L_X, \mathcal{Q}(g)] = \frac{\hbar}{\mathrm{i}}\mathcal{Q}(\{J_X, g\})$ for all $X \in \mathfrak{g}$.

Generalization to $G$-bundles $E, F$,

$$\mathcal{Q} : \mathcal{S}(M) \otimes \Gamma(E^* \otimes F) \to \mathcal{D}(M; E, F).$$

Most studied examples : $G = \mathrm{SL}(n+1), \mathrm{O}(p+1, q+1)$ with density bundles $|\Lambda T^*M|^{\otimes\lambda}$.

We suppose that $(M, g)$ is a spin manifold, of dimension $2n$ and signature $(p, q)$.

| classical | quantum |
|---|---|
| supercotangent $(\mathcal{M}, \omega)$ | spinors $\mathcal{H} = \mathrm{L}^2(\mathsf{S})$ |
| refined symbols $\mathcal{S}(M) \otimes \Omega_{\mathbb{C}}(M)$ | spinor diff. Op. $\mathrm{D}(M, \mathsf{S})$ |
| Hamiltonian lift of $\mathrm{conf}(M, g)$ | $\mathrm{ad}$(Lie derivative of spinors) |

1. Usual symbol space $\mathcal{S}(M) \otimes \Gamma(\mathbb{C}\mathrm{l}(M, g))$.
2. Complex functions of $\mathcal{M} = T^*M \times_M \Pi TM$ form the algebra $\mathcal{C}^\infty(T^*M) \otimes \Omega_{\mathbb{C}}(M)$, generated by $(x^i, p_i, \xi^i)$.
3. For $X \in \mathrm{conf}(M, g)$, the natural lift $\hat{X}$ differs from the Hamiltonian lift $\tilde{X}$.

The momentum of a rotation is :

$$\mathcal{J}_{X_{ij}} = p_i x_j - x_j p_i + \frac{\hbar}{\mathrm{i}} \xi_i \xi_j,$$

hence the spin is $S_{ij} = \frac{\hbar}{\mathrm{i}} \xi_i \xi_j$, generating $(\Omega^2_x(M), \{\cdot, \cdot\}) \simeq \mathrm{o}(p, q)$.

Example

Rotating particle e.o.m.$=$ Papapetrou equations $=$ Hamiltonian flows of $g^{ij} p_i p_j$ :

$$\dot{x}^j \nabla_j \dot{x}^i = -\frac{1}{2} g^{ik} R(S)_{jk} \dot{x}^j,$$
$$\dot{x}^k \nabla_k S^{ij} = 0.$$

# Quantization of $(\mathcal{M}, \omega)$

Geometric quantization of $(\mathcal{M}, \omega)$ (need topological restrictions on $M$) gives a way to construct [M. '10]

1. the spinor bundle S as quantum space,
2. the Lie derivative of spinors $\mathrm{L}_X = \mathcal{Q}_G(\mathcal{J}_X)$,
3. the covariant derivative of spinors $\nabla_X = \mathcal{Q}_G(p_i X^i)$,
4. generators of the Clifford bundle $\gamma^i = \mathcal{Q}_G(\xi^i)$.

Aim : extend $\mathcal{Q}_G$ by the conformally equivariant quantization :

$$\mathcal{Q} : \mathcal{S}(M) \otimes \Omega_{\mathbb{C}}(M) \longrightarrow \mathrm{D}(M, \mathrm{S}),$$

equivariant w.r.t. the action of $X \in \mathrm{conf}(M, g)$,

$$\mathcal{Q}(\tilde{X} f) = \mathrm{L}_X \mathcal{Q}(f) - \mathcal{Q}(f) \mathrm{L}_X.$$

# Three $\mathrm{o}(p+1, q+1)$-modules

From now on, $(M, g)$ is a conformally flat manifold, locally $\mathrm{conf}(M, g) \simeq \mathrm{o}(p+1, q+1)$. We define three $\mathrm{o}(p+1, q+1)$-modules :

1. $\mathrm{D}^{\lambda,\mu}$, of spinor diff. op. $A : \Gamma(\mathrm{S} \otimes |\Lambda T^*M|^{\otimes\lambda}) \to \Gamma(\mathrm{S} \otimes |\Lambda T^*M|^{\otimes\mu})$, with adjoint action $\mathcal{L}_X^{\lambda,\mu}$ ;
2. $\mathcal{S}^\delta[\xi]$, of Hamiltonian symbols $P \in \mathcal{S}(M) \otimes \Omega_{\mathbb{C}}(M) \otimes \Gamma(|\Lambda T^*M|^{\otimes\delta})$, with Hamiltonian action $L_X^\delta = \tilde{X} + \delta \mathrm{Div}(X)$ ;
3. $\mathcal{T}^\delta[\xi]$, of tensor symbols $P \in \bigoplus_{\kappa=0}^n \mathcal{S}(M) \otimes \Omega_{\mathbb{C}}^\kappa \otimes \Gamma(|\Lambda T^*M|^{\otimes(\delta - \frac{\kappa}{n})}$, with natural action $L_X^\delta = \hat{X} + (\delta - \frac{\kappa}{n})\mathrm{Div}(X)$.

$\mathcal{T}^\delta[\xi]$ is isomorphic to $\mathcal{S}(M) \otimes \mathbb{C}\mathrm{l}(M, g) \otimes \Gamma(|\Lambda T^*M|^{\otimes\delta})$ and satisfies

$$\mathcal{T}^\delta[\xi] \simeq \mathrm{Gr}\mathcal{S}^\delta[\xi] \simeq \mathrm{Gr}\mathrm{D}^{\lambda,\mu},$$

if $\delta = \mu - \lambda$.

# Main result

**Theorem**

*There exists $I^u_{\mathrm{S}} \sqcup I^e_{\mathrm{S}} = I_{\mathrm{S}} \subset \mathbb{Q}^*_+$ such that the conformally equivariant superization*

$$\mathrm{S}^\delta_{\mathcal{T}} : \mathcal{T}^\delta[\xi] \to \mathcal{S}^\delta[\xi],$$

*exists if $\delta \notin I^e_{\mathrm{S}}$ and is unique if $\delta \notin I_{\mathrm{S}}$.*

**Theorem**

*There exists $I^u_{\mathcal{Q}} \sqcup I^e_{\mathcal{Q}} = I_{\mathcal{Q}} \subset \mathbb{Q}^*_+$ such that the conformally equivariant quantization*

$$\mathcal{Q}^{\lambda,\mu} : \mathcal{S}^\delta[\xi] \to \mathrm{D}^{\lambda,\mu},$$

*exists if $\delta \notin I^e_{\mathcal{Q}}$ and is unique if $\delta \notin I_{\mathcal{Q}}$.*

Remark : $\mathcal{Q}^{\lambda,\mu} \circ \mathrm{S}^\delta_{\mathcal{T}}$ is a particular case of AHS-equivariant quantization [Cap, Silhan '10].

# Scheme of proof and explicit construction

## Preliminaries

1. local identification $\mathcal{T}^\delta[\xi] \simeq \mathcal{S}^\delta[\xi] \simeq \mathrm{D}^{\lambda,\mu}$ as $\mathrm{ce}(p,q)$-modules ;
2. determination of $\mathrm{ce}(p,q)^!$, the algebra of $\mathrm{ce}(p,q)$-invariant in $\mathrm{End}(\mathcal{T}^\delta[\xi])$ ;
3. computation of the three Casimir operators $C_\mathcal{T}$, $C_\mathcal{S}$ and $C_\mathrm{D}$.

## Necessary condition :

$$\mathrm{S}^\delta_\mathcal{T} C_\mathcal{T} = C_\mathcal{S} \mathrm{S}^\delta_\mathcal{T} \quad \text{and} \quad \mathcal{Q}^{\lambda,\mu} C_\mathcal{S} = C_\mathrm{D} \mathcal{Q}^{\lambda,\mu}.$$

## Key idea :

$$\text{(eigenvector)} \longmapsto \text{(eigenvector with same principal symbol)}\,.$$

1. $C_\mathcal{S} - C_\mathcal{T}$ and $C_\mathrm{D} - C_\mathcal{S}$ are strictly lowering the degree in $p$ ;
2. $C_\mathcal{T}$ admits a basis of eigenvectors in $\mathcal{T}^\delta[\xi]$.

# Isometric invariants of $\mathrm{End}_{\mathrm{Diff}}(\mathcal{T}^\delta[\xi])$

Let $\mathrm{e}(p,q)^!$ be the subalgebra of local isometric invariants in $\mathrm{End}_{\mathrm{Diff}}(\mathcal{T}^\delta[\xi])$. Example of Howe dual pairs [Howe '76].

## Proposition

*$\mathrm{e}(p,q)^!$ is isomorphic to $\mathfrak{U}(\mathrm{spo}(2|2) \ltimes \mathfrak{h}(2|2))$, and generated by*

$$R = \eta^{ij} p_i p_j, \quad \mathcal{E}_n = p_i \partial_{p_i} + \frac{n}{2}, \quad T = \eta^{ij} \partial_{p_i} \partial_{p_j}, \quad \Sigma_n = \xi^i \partial_{\xi^i} - \frac{n}{2},$$

$$\Delta = \xi^i p_i, \quad \Phi = \eta^{ij} p_i \partial_{\xi^j}, \quad \Psi = \eta_{ij} \xi^i \partial_{p^j}, \quad \Omega = \partial_{\xi^i} \partial_{p_i},$$

*generating the Lie algebra $\mathrm{spo}(2|2)$, and by the operators*

$$G = \eta^{ij} p_i \partial_j, \quad D = \partial_{p_i} \partial_i, \quad L = \eta^{ij} \partial_i \partial_j,$$

$$\Gamma = \xi^i \partial_i, \quad \Lambda = \eta^{ij} \partial_{\xi^i} \partial_j,$$

*generating the Lie algebra $\mathfrak{h}(2|2)$.*

# Diagonalization of the Casimir operator $C_{\mathcal{T}}$

$$C_{\mathcal{T}} = RT + 2(\Delta\Omega + \Phi\Psi) + \Sigma(\Sigma - n) + 2\,[n(\delta - 1) - \mathcal{E}]\,\mathcal{E} - n^2\delta(\delta - 1),$$

$\approx$ Casimir operator of $\mathrm{o}(p,q)$ on the fibers of $\mathcal{M}$.

$$(\text{diag. of } C_{\mathcal{T}}) \longleftrightarrow (\text{ Irr. decompo. of } \mathcal{S}T_xM \times \Lambda T^*_xM \text{ w.r.t. } \mathrm{o}(p,q))$$

1. $\mathcal{E}$, $RT$ : harmonic spheric $\mathcal{S}(M) = \bigoplus_{k,s} R^s(\mathcal{S}_k(M) \cap \ker T)$,
2. $\Sigma$ : graduation $\Omega(M) = \bigoplus_{\kappa} \Omega^{\kappa}(M)$,
3. $\Delta\Omega$ and $\Phi\Psi$ are non-commuting projectors.

Introducing $\Delta_0 = (\mathrm{Id} - P(\mathcal{E})RT)\Delta$, we obtain the eigenspaces

$$\mathcal{T}^{\delta}_{k,\kappa,s;\,ab}[\xi] = R^s(\Delta_0)^a\Phi^b\left(\mathcal{T}^{\delta}_{k',\kappa'}[\xi] \cap (\ker T \cap \ker\Omega \cap \ker\Psi)\right),$$

whose fiber in one point is irreducible w.r.t. $\mathrm{o}(p,q)$, we have $k' = k - 2s - a + b$ and $\kappa' = \kappa - a + b$.

# Examples and applications

1. We get explicit formulae for $\mathrm{S}^{\delta}_{\mathcal{T}}$ and $\mathcal{Q}^{\lambda,\mu}$ restricted to degree 1 symbols,

$$J_X \overset{\mathrm{S}^{\delta}_{\mathcal{T}}}{\longmapsto} \mathcal{J}_X \overset{\mathcal{Q}^{\lambda,\lambda}}{\longmapsto} \mathrm{L}^{\lambda}_X,$$

$$\{p_i\xi^i, \mathrm{S}^{\delta}_{\mathcal{T}}(T)\} = 0 \;\Leftrightarrow\; T \text{ is a Killing-Yano tensor.}$$

# Examples and applications

1. We get explicit formulae for $\mathrm{S}^{\delta}_{\mathcal{T}}$ and $\mathcal{Q}^{\lambda,\mu}$ restricted to degree 1 symbols,

$$J_X \overset{\mathrm{S}^{\delta}_{\mathcal{T}}}{\longmapsto} \mathcal{J}_X \overset{\mathcal{Q}^{\lambda,\lambda}}{\longmapsto} \mathrm{L}^{\lambda}_X,$$

$$\{p_i\xi^i, \mathrm{S}^{\delta}_{\mathcal{T}}(T)\} = 0 \iff T \text{ is a Killing-Yano tensor.}$$

2. Let $R = g^{ij}p_ip_j$ and $\Delta = p_i\xi^i$, globally defined on $(M, g)$.

**Theorem**

*The conformal invariants of each families of modules are*

1. $\Delta^a R^s \in \mathcal{T}^{\frac{2s+a}{n}}[\xi]$, *where* $s \in \mathbb{N}$ *and* $a = 0, 1$.
2. $\Delta R^s \in \mathcal{S}^{\frac{2s+a}{n}}[\xi]$, *where* $s \in \mathbb{N}$.
3. $\mathcal{Q}^{\lambda,\mu}(\Delta R^s) \in \mathrm{D}^{\frac{n-2s-1}{2n}, \frac{n+2s+1}{2n}}$, *for all* $s \in \mathbb{N}$.

# Conformal invariants and equivariant maps

If $\mathrm{S}_{\mathcal{T}}^{\delta}$ and $\mathcal{Q}^{\lambda,\mu}$ exist, the conformal invariants correspond to each other via

$$\mathcal{T}^{\delta}[\xi] \xrightarrow{\mathrm{S}_{\mathcal{T}}^{\delta}} \mathcal{S}^{\delta}[\xi] \xrightarrow{\mathcal{Q}^{\lambda,\mu}} \mathsf{D}^{\lambda,\mu}$$

We deduce then

1. non-existence of $\mathrm{S}_{\mathcal{T}}^{\delta}$ for $\delta = \frac{2s}{n}$ ;
2. if $\delta = \frac{2s+1}{n}$, $\mathcal{Q}^{\lambda,\mu}$ exists iff $\lambda = \frac{1-\delta}{2}$,
3. the module $\mathsf{D}^{\lambda,\mu}$ is exceptionnal among $\mathsf{D}^{\sigma,\sigma+\delta}$, with $\sigma \in \mathbb{R}$.

# Thanks !

References :

*Quantification conformément équivariante des fibrés supercotangents*, Jean-Philippe Michel, thèse, tel-00425576 version 1.

*Conformal geometry of the supercotangent and spinor bundles*, Jean-Philippe Michel, arXiv :1004.1595v1.

*Conformally equivariant quantization of supercotangent bundles*, Jean-Philippe Michel, in preparation.