Higher symmetries of Laplacian via quantization

Jean-Philippe Michel

Mathematics Research Unit,
University of Luxembourg

Naive definition of higher symmetries of Laplacian

On pseudo-Euclidean space : $\mathbb{R}^{p,q}$, $\eta = \mathrm{Id}_p \oplus (-\mathrm{Id}_q)$ and $p + q = n$, the Laplacian is given by

$$\Delta = \eta^{ij}\partial_i\partial_j.$$

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Its higher symmetries are differential operators D_1 s.t.

$$\exists D', [\Delta, D_1] = D'\Delta \qquad \text{or} \qquad \exists D_2, \Delta D_1 = D_2\Delta.$$

Example : first order higher symmetries are given by

$$\Delta(X + \lambda \mathrm{Div} X) = (X + \mu \mathrm{Div} X)\Delta,$$

where $L_X\eta = F\eta$, i.e. $X \in \mathrm{o}(p+1, q+1) =: \mathfrak{g}$, and $\lambda = \frac{n-2}{2n}$, $\mu = \frac{n+2}{2n}$.

Remark : the space of HS is an algebra and a $\mathfrak{g}$-module.

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$$\begin{array}{rcl} \text{Coadjoint orbits of } G\,(=\mathrm{O}(p+1,q+1)) & \longrightarrow & \text{UIR of } G \\ \text{Minimal (nilpotent) coadjoint orbit } \mathcal{O}_{\mathrm{min}} & \longrightarrow & ? \end{array}$$

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Joseph '76 : $\exists!$ primitive ideal J s.t. $\mathrm{gr}(\mathfrak{U}(\mathfrak{g})/J) \simeq \mathrm{Poly}[\mathcal{O}_{\text{min}}]$.
Binegar and Zierau '91 : $\mathfrak{U}(\mathfrak{g})$ acts on ker Δ with kernel J. If $p+q$ even, it integrates in a UIR of G, the minimal representation.

$$J' = J\,!$$

Aims

- To propose a new method to classify the HS of Δ and to determine the algebra structure of the space of HS.
  - Eastwood ('02), and Leistner ('06) : conformal ambient space,
  - Gover and Silhan ('09) : tractor techniques,
  - here : quantization and symplectic reduction.
- To provide a geometrical link between $\ker\Delta$ and the Joseph ideal, via the minimal nilpotent coadjoint orbit.

Defnition of HS of Laplacian

Geometric setting :

(M, g) conformally flat manifold,

λ-densities $\Gamma(|\Lambda^n T^*M|^{\otimes\lambda}) \simeq (\mathcal{C}^\infty(M), \ell^\lambda)$ with $\ell^\lambda_X = X + \lambda \mathrm{Div} X$,

$$\Delta \ell^\lambda_X = \ell^\mu_X \Delta,$$

for $X \in \mathfrak{g}$, $\lambda = \frac{n-2}{2n}$, $\mu = \frac{n+2}{2n}$, and $\Delta \in \mathcal{D}^{\lambda,\mu}$ the conformal Laplacian.

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Trivial symmetries : $\Delta(P\Delta) = (\Delta P)\Delta$, i.e. $(\Delta) = \{P\Delta, P \in \mathcal{D}^{\mu,\lambda}\}$.

Definition

The algebra of HS of Δ is $\mathcal{A}^{\lambda,1} \subset \mathcal{D}^{\lambda,\lambda}/(\Delta)$, and $[D_1] \in \mathcal{A}^{\lambda,1}$ satisfies

$$\exists D_2 \in \mathcal{D}^{\mu,\mu}, \ \textit{s.t.}\ \Delta D_1 = D_2 \Delta.$$

$$\mathcal{A}^{\lambda,1} = \ker \mathrm{QHS} : \begin{array}{ccc} \mathcal{D}^{\lambda,\lambda}/(\Delta) & \to & \mathcal{D}^{\lambda,\mu}/(\Delta) \\ [D_1] & \mapsto & [\Delta D_1] \end{array} \qquad \text{conf. inv.}$$

Conformal Killing tensors

Principal symbol map ($\delta = \mu - \lambda$) :

$$\sigma : \mathcal{D}_k^{\lambda,\mu} \longrightarrow \mathrm{Pol}_k^{\delta}(T^*M).$$

Example : $H := \sigma(\Delta) = \mathrm{g}^{ij}p_ip_j$, its Hamiltonian flow project on the geodesic flow on (M, g).

The map σ satisfies $\sigma([A, B]) = \{\sigma(A), \sigma(B)\}$, hence (on $\mathbb{R}^{p,q}$)

$$[\Delta, D_1] = D'\Delta \Rightarrow \{H, \sigma(D_1)\} = \sigma(D')H.$$

It means $\sigma(D_1)$ is a conformal Killing tensor.

Definition

The space of symmetries of the null geodesic flow is $\mathcal{K}^1 = \{$*Traceless CKT*$\}$*, a subalgebra of* $\mathrm{Pol}(T^*M)/(H)$.

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We have $\sigma : \mathcal{A}_k^{\lambda,1} \longrightarrow \mathcal{K}_k^1$, does it exist a section ?

More generally, does it exist a $\mathfrak{g}$-equivariant section to σ ?

Conformally equivariant quantization

Theorem (Duval, Lecomte, Ovsienko '99)

Let (M, g) *conformally flat manifold. For every* $k \in \mathbb{N}$ *and (generic) shift* $\delta = \mu - \lambda$*,*

$$\exists! \mathcal{Q}^{\lambda,\mu} : \mathrm{Pol}_k^\delta(T^*M) \to \mathcal{D}_k^{\lambda,\mu} \qquad \textit{s.t.}$$

(i) $\mathcal{Q}^{\lambda,\mu}$ *is a right inverse the principal symbol map,* $\sigma \circ \mathcal{Q}^{\lambda,\mu}_{|\mathrm{Pol}_k} = \mathrm{Id}$*,*
(ii) $\mathcal{Q}^{\lambda,\mu}$ *intertwines the* $\mathfrak{g}$*-action.*

Equivariant quantizations exist for various locally flat geometries (IFFT or $|1|$-graded) and differential operators acting on natural vector bundles. They admit curved analog in terms of Cartan geometries [Mathonet, Radoux '05-'08] and [Cap, Silhan '09].

Explicit formulae are known for $\mathcal{Q}^{\lambda,\mu}$.

Classification of HS of Laplacian

Theorem (Eastwood '02)

For $\lambda = \frac{n-2}{2n}$, we have the isomorphism of $\mathfrak{g}$-modules $\mathcal{Q}^{\lambda,\lambda} : \mathcal{K}^1 \to \mathcal{A}^{\lambda,1}$. Moreover, every $P \in \mathcal{K}^1$ satisfies $\Delta \mathcal{Q}^{\lambda,\lambda}(P) = \mathcal{Q}^{\mu,\mu}(P)\Delta$.

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Idea of proof

$$\begin{array}{ccc}
\mathcal{D}^{\lambda,\lambda}/(\Delta) & \xrightarrow{\text{QHS}} & \mathcal{D}^{\lambda,\mu}/(\Delta) \\
\uparrow{\scriptstyle \mathcal{Q}^{\lambda,\lambda}} & & \uparrow{\scriptstyle \mathcal{Q}^{\lambda,\mu}} \\
\mathrm{Pol}^{0}_{*,0}(T^*M) & \xrightarrow[?]{} & \mathrm{Pol}^{\mu-\lambda}_{*,0}(T^*M)
\end{array}$$

we identify ? thanks to the classification of conformally invariant operators. Its kernel is $\mathcal{K}^1$.

Interpretation : classical symmetries $\leftrightarrow$ quantum symmetries.

The algebra structure of HS (I)

Let $\mathcal{K}$ be the algebra generated by $\mathcal{K}^1$ in $\mathrm{Pol}(T^*M)$, and $\mathcal{A}^\lambda := \mathcal{Q}^{\lambda,\lambda}(\mathcal{K})$.

Theorem

We get the following commutative diagram

$$\begin{array}{ccc} \mathrm{S}(\mathfrak{g}) & \xrightarrow{\Phi^\lambda} & \mathfrak{U}(\mathfrak{g}) \\ \downarrow{\scriptstyle \mu^*} & & \downarrow{\scriptstyle \ell^\lambda} \\ \mathrm{S}(\mathfrak{g})/I \simeq \mathcal{K} & \xrightarrow[\mathcal{Q}^{\lambda,\lambda}]{} & \mathcal{A}^\lambda \simeq \mathfrak{U}(\mathfrak{g})/J^\lambda \end{array}$$

with $\Phi^\lambda = \mathrm{Sym} \circ \phi^\lambda$ *and* $\phi^\lambda = \mathrm{Id}_{\mathrm{S}(\mathfrak{g})} + N$ *where* N *lowers the degree.*

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S(\mathfrak{g})/I \simeq \mathcal{K} & \xrightarrow[\mathcal{Q}^{\lambda,\lambda}]{} & \mathcal{A}^\lambda \simeq \mathfrak{U}(\mathfrak{g})/J^\lambda \\
\downarrow & & \downarrow \\
\mathcal{K}^1 \simeq \mathcal{K}/(H) & \xrightarrow[\mathcal{Q}^{\lambda,\lambda}]{} & \mathcal{A}^\lambda/(\Delta) \simeq \mathcal{A}^{\lambda,1}
\end{array}$$

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Coadjoint orbits of G via symplectic reduction

We restrict to $M = \mathbb{S}^p \times \mathbb{S}^q \subset \mathbb{R}^{p+1,q+1}$, and $p, q \geq 1$, $n = p + q \geq 3$.
Recall that $\mathfrak{g}^* \simeq \Lambda^2\mathbb{R}^{p+1,q+1}$. We have the following momentum map

$$\begin{array}{ccccc} T^*\mathbb{R}^{p+1,q+1} & \xrightarrow{\mathrm{SL}(2,\mathbb{R})} & \mathrm{Bv} & \xrightarrow{\mathbb{R}^*} & \mathrm{Gr}(2, n+2) \cup \{0\} \\ (u, v) & \mapsto & u \wedge v & \mapsto & \mathrm{span}(u, v) \end{array}$$

Fact : $\mathcal{O}_{min} \overset{\mathbb{R}^*}{\to} P(0, 0)$.

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Fact : $\mathcal{O}_{min} \xrightarrow{\mathbb{R}^*} P(0, 0)$.
$(G, \mathrm{SL}(2, \mathbb{R}))$ is a Howe dual pair in $\mathrm{Sp}(2n + 2, \mathbb{R})$. In $T^*\mathbb{R}^{p+1,q+1}$, we get $\mathrm{sl}(2, \mathbb{R}) = \langle x^2, xp, p^2 \rangle$.

Theorem

$$\begin{array}{ccc} T^*(\mathbb{R}^{p+1,q+1} \setminus \{0\}) // \left\langle x^2, xp \right\rangle & \xrightarrow{\simeq} & T^*M \\ (T^*M \setminus M) // \langle H \rangle & \xrightarrow{\mathbb{Z}_2} & \mathcal{O}_{min} \end{array}$$

and we have $\mathcal{K} \simeq \mathrm{Poly}[T^*_\pm M]$, $\mathcal{K}^1 \simeq \mathrm{Poly}[\mathcal{O}_{min}]$.

Joseph ideal

Corollary

$\mathcal{A}^{\lambda,1} \simeq \mathfrak{U}(\mathfrak{g})/J^{\lambda,1}$ *with* $J^{\lambda,1}$ *is the Joseph ideal, hence the representation of* $\mathfrak{g}$ *on* $\ker \Delta$ *via* ℓ^{λ} *is minimal.* $\mathcal{Q}^{\lambda,\lambda} : \mathcal{K}^1 \to \mathcal{A}^{\lambda,1}$ *is a quantization of* $\mathcal{O}_{min}$.

The algebra structure of HS (II)

Recall that $\mathfrak{g} \simeq \Lambda^2\mathbb{R}^{p+1,q+1} = \ydiagram{1,1}$. We have

$$\mathfrak{g} \odot \mathfrak{g} = \ydiagram{2,2} \oplus \ydiagram{1,1,1,1} \quad \text{and} \quad \ydiagram{2,2} = \ydiagram{2,2}_0 \oplus \ydiagram{2}_0 \oplus C\mathbb{R}.$$

The morphism $\mathrm{S}(\mathfrak{g}) \to \mathrm{Pol}(T^*\mathbb{R}^{p+1,q+1})$ has kernel $\left(\ydiagram{1,1,1,1}\right)$. Moreover, the Casimir writes on $T^*\mathbb{R}^{p+1,q+1}$: $C = (xp)^2 - x^2p^2$.

Theorem

We obtain $I = \left(\ydiagram{1,1,1,1}\right) + (C)$ *and* $I^1 = I + (\ydiagram{2}_0)$.

Via $\Phi^\lambda = \mathrm{Sym} \circ \phi^\lambda$, *we get*

$$J^\lambda = \Big(\mathrm{Sym}\left(\ydiagram{1,1,1,1}\right) + \mathrm{Sym}(C) - a(\lambda)\Big) \text{ and } J^{\lambda,1} = J^\lambda + \left(\mathrm{Sym}\left(\ydiagram{2}_0\right)\right),$$

where $a(\lambda)$ *is the eigenvalue of the Casimir operator on* λ*-densities.*

Thanks !