Quantization and conformal geometry of the supercotangent and spinor bundles

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Quantization = Correspondence between classical and quantum mechanics

	classical	quantum
Phase space	$(\mathcal{M},\omega)$	$\mathcal{H}$
Observables	$A \subset \mathcal{C}^\infty(\mathcal{M})$	$\mathcal{A} \subset \mathcal{L}(\mathcal{H})$
Symmetry	$\mathfrak{g} \subset \mathrm{ham}(\mathcal{M},\omega)$	$\mathfrak{g} \subset \mathrm{u}(\mathcal{H})$.

1. Classical space : graded algebra $A = \bigoplus_{k=0}^{\infty} A_k$ s.t. $A_k \cdot A_l \subset A_{k+l}$.
2. Quantum space : filtered algebra $\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_k$ s.t. $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots$ and $\mathcal{A}_k \cdot \mathcal{A}_l \subset \mathcal{A}_{k+l}$.
3. Link : $\mathrm{Gr}\,\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k/\mathcal{A}_{k-1}$.

Example

Let M be the configuration space of a physical system (without spin).

	classical	quantum
Phase space	T^*M	$\mathrm{L}^2(M)$
Observables	Symbols $\mathcal{S}(M)$	Diff. Op. $\mathcal{D}(M)$

Quantization $=$ (symbol map)$^{-1}$,

A Lie algebra $\mathfrak{g} \subset \mathrm{Vect}(M)$ acts canonically on $\mathcal{S}(M)$ and $\mathcal{D}(M)$.

Example : on (M, g), locally, $\mathrm{conf}(M, g) = \{X \in \mathrm{Vect}(M) | L_X g_{ij} = \lambda g_{ij}\}$. This is of maximal dimension if (M, g) is conformally flat, i.e. $g_{ij} = F\eta_{ij}$ locally.

Problematics

Let M be the configuration manifold of a spin system.
We suppose that (M, g) is a spin manifold, of dimension $2n$ and signature (p, q). We denote by S its spinor bundle.

classical	quantum
supercotangent $(\mathcal{M}, \omega)$	spinors $\mathcal{H} = \mathrm{L}^2(\mathrm{S})$
symbols $\mathcal{S}(M)[\xi]$	spinor diff. Op. $\mathrm{D}(M, \mathrm{S})$
$\mathrm{conf}(M, g) \overset{?}{\hookrightarrow} \mathrm{ham}(\mathcal{M}, \omega)$	$\mathrm{conf}(M, g) \overset{?}{\hookrightarrow} \mathrm{U}(\mathcal{H})$

1. Actions of conformal vector fields of (M, g)
2. Geometric quantization of the supercotangent
3. Classification of the conformally covariant elements
4. Conformally equivariant quantization, defined on $\mathcal{S}(M)[\xi]$.

From spin geometry to supergeometry

Over one point :

classical	quantum
? ?	spinor module S
$\mathrm{Gr}\,\mathbb{C}\mathrm{l}(V^*, g) \simeq \Lambda V^* \otimes \mathbb{C}$	$\mathbb{C}\mathrm{l}(V^*, g) \simeq \mathrm{End}(S)$

Over (M, g) :

- Clifford bundle $\mathrm{Cl}(M, g)$, spin bundle S and $\mathrm{Gr}\,\Gamma(\mathrm{Cl}(M, g)) = \Omega(M)$.
- Differential operators acting on $\Gamma(\mathrm{S})$: $\mathcal{D}(M, \mathrm{S}) = \mathcal{D}(M) \otimes \Gamma(\mathbb{C}\mathrm{l}(M, g))$.
- Symbols : $\mathcal{S}(M)[\xi] = \mathcal{S}(M) \otimes \Omega_{\mathbb{C}}(M)$.

Supercommutative algebra : $ab = (-1)^{|a||b|} ba$.
Definition : Let $E \to M$ be a vector bundle, it defines the supermanifold $\Pi E = (M, \Gamma(\cdot, \Lambda E^*))$, with space of functions $\mathcal{C}^\infty(\Pi E) = \Gamma(M, \Lambda E^*)$ and coordinates (x^i, ξ^a).
Examples : $\mathcal{C}^\infty(\Pi V) = \Lambda V^*$ and $\mathcal{C}^\infty(\Pi TM) = \Omega(M)$.

The supercotangent bundle

The supercotangent bundle is the supermanifold

$$\mathcal{M} = T^*M \times_M \Pi TM,$$

whose space of functions is

$$\mathcal{C}^\infty(\mathcal{M}) = \mathcal{C}^\infty(T^*M) \otimes \Omega(M),$$

generates by the local coordinates (x^i, p_i, ξ^i). It contains $\mathcal{S}(M)[\xi]$.

There is a correspondence between (M, g, ∇) and $(\mathcal{M}, \omega)$, where $\omega = d\alpha$ is symplectic and

$$\alpha = p_i dx^i + \frac{\hbar}{2\mathrm{i}} g_{ij} \xi^i d^\nabla \xi^j.$$

Hamiltonian actions on $(\mathcal{M}, \omega)$

Remark : the natural lift of X : $L(X) = X^i\partial_i - p_j\partial_i X^j\partial_{p_i} + \xi^i\partial_i X^j\partial_{\xi^j}$, does not preserve α.

Proposition

The condition $L_{\hat{X}}\alpha = 0$ does not fix a lift $\hat{X}$ of $X \in \mathrm{Vect}(M)$ to $\mathcal{M}$.

We introduce $\beta = g_{ij}\xi^i dx^j$, the pull-back of the canonical 1-form of ΠTM.

Theorem

Only the vector fields $X \in \mathrm{conf}(M, g)$ admit a lift $\widetilde{X}$ preserving α and the direction of β. This lift is unique.

Denoting by ev_g :natural coord. $\mapsto$ Darboux coord., we have

$$\tilde{X} = \mathrm{ev}_g L(X)(\mathrm{ev}_g)^{-1} + \text{nilpotent}$$

Geometric quantization of the supercotangent $(\mathcal{M}, \omega)$

Starting from $(\mathcal{M}, \omega)$ and a polarization, the geometric quantization construct

1. a quantum representation space $\mathcal{H}$,
2. a Lie algebra morphism $\mathcal{Q}_{QG} : \mathrm{Obs} \subset \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{L}(\mathcal{H})$.

From $N \subset T_{\mathbb{C}}M$ an isotropic maximal subbundle of $T_{\mathbb{C}}M$ for g, we define a polarization of $\mathcal{M}$.

Proposition

Geometric quantization proves that ΛN^* is the spinor bundle of M, and $\mathcal{Q}_{QG} : \mathrm{Obs} \subset \mathcal{S}(M)[\xi] \to \mathcal{D}(M, \mathrm{S})$ is a Lie algebra morphism s.t. $\mathcal{Q}_{QG}(p_i) = \frac{\hbar}{\mathrm{i}} \nabla_i$ and $\mathcal{Q}_{QG}(\xi^i) = \frac{\gamma^i}{\sqrt{2}}$.

Spinor Lie derivatives

The comoment map of $\mathrm{conf}(M, g)$ on $\mathcal{M}$ is the Lie algebra morphism $\mathcal{J} : \mathrm{conf}(M, g) \to \mathcal{C}^\infty(\mathcal{M})$ given by $\mathcal{J}_X = \left\langle \tilde{X}, \alpha \right\rangle$.

Proposition

Let $X \in \mathrm{conf}(M, g)$, we have

$$\mathcal{Q}_{QG}(\mathcal{J}_X) = \frac{\hbar}{\mathrm{i}} \mathsf{L}_X$$

where L is the Lie derivative of spinors, introduced by Kosmann.

The $\mathrm{conf}(M,g)$-modules $\mathcal{S}^\delta[\xi]$ et $\mathsf{D}^{\lambda,\mu}$

The structure of $\mathrm{conf}(M,g)$-module of $\mathsf{F}^\lambda = \Gamma(\mathsf{S}) \otimes \Gamma(|\Lambda T^*M|^{\otimes\lambda})$, the space of λ-spinor densities, is given by

$$\mathsf{L}_X^\lambda = \mathsf{L}_X + \lambda \mathrm{Div}(X).$$

We introduce $\mathsf{D}^{\lambda,\mu}$ the module of differential operators between F^λ and F^μ, endowed with the action of $\mathrm{conf}(M,g)$,

$$\mathcal{L}_X^{\lambda,\mu} A = \mathsf{L}_X^\mu A - A \mathsf{L}_X^\lambda.$$

The space of symbols is $\mathcal{S}^\delta[\xi] = \mathcal{S}(M)[\xi] \otimes \Gamma(|\Lambda T^*M|^{\otimes\delta})$, where $\delta = \mu - \lambda$. It admits the following action of $\mathrm{conf}(M,g)$,

$$L_X^\delta = \widetilde{X} + \delta \mathrm{Div}(X).$$

Remark : using the normal ordering $\mathcal{N} : (x^i, p_i, \xi^i) \mapsto \left(x^i, \frac{\hbar}{\mathrm{i}}\nabla_i, \frac{\gamma^i}{\sqrt{2}}\right)$,

we get

$$\mathcal{N}^{-1}\mathcal{L}_X^{\lambda,\mu}\mathcal{N} = L_X^\delta + \text{nilpotent}$$

The isometric invariants

We suppose (M, g) conformally flat, and we denote by $\mathfrak{e}(p, q)$ the subalgebra of isometries.

If $X \in \mathfrak{e}(p, q)$, we have

$$\mathcal{N}^{-1}\mathcal{L}_X^{\lambda,\mu}\mathcal{N} = L_X^{\delta} = \mathrm{ev}_g L(X)(\mathrm{ev}_g)^{-1}.$$

Let ε be the canonical volum form of $\mathbb{R}^n$, and $(x^i, \tilde{p}_i, \tilde{\xi}^i)$ be Darboux coordinates on $(\mathcal{M}, \omega)$.

Proposition

The subalgebra of isometric invariants of $\mathcal{S}^\delta[\xi]$ is generated by

$$R = \eta^{ij}\tilde{p}_i\tilde{p}_j, \quad \Delta = \tilde{\xi}^i\tilde{p}_i, \quad \chi = \varepsilon_{j_1\ldots j_n}\tilde{\xi}^{j_1}\ldots\tilde{\xi}^{j_n} \text{ et } \Delta{*}\chi = \varepsilon_{j_1\ldots j_n}\tilde{p}^{j_1}\tilde{\xi}^{j_2}\ldots\tilde{\xi}^{j_n}.$$

Classification of conformal invariants

The scalar conformal invariants are $R^k \in \mathcal{S}^{\frac{2k}{n}}$ and $\mathcal{N}(R^k) \in \mathcal{D}^{\frac{n-2k}{2n},\frac{n+2k}{2n}}$.

Theorem

The conformal invariants are given by

1. $\Delta^a * \chi^b R^s \in \mathcal{S}^{\frac{2s+a}{n}}[\xi]$, *where* $s \in \mathbb{N}$ *and* $a, b = 0, 1$ *with* $a + b \neq 0$.
2. $\mathcal{N}(\chi) \in \mathrm{D}^{\lambda,\lambda}$, $\mathcal{N}(\Delta * \chi) \in \mathrm{D}^{\frac{n-1}{2n},\frac{n+1}{2n}}$, *and* $\mathcal{N}(\Delta R^s) \in \mathrm{D}^{\frac{n-2s-1}{2n},\frac{n+2s+1}{2n}}$, *for all* $\lambda \in \mathbb{R}$ *and* $s \in \mathbb{N}$.

Remark : the conformally invariants of $\mathrm{D}^{\lambda,\mu}$ are then

- the chirality : $(\mathrm{vol}_g)_{i_1\cdots i_n}\gamma^{i_1}\ldots\gamma^{i_n} \in \mathrm{D}^{\lambda,\lambda}$,
- the Dirac operator : $\gamma^i\nabla_i \in \mathrm{D}^{\frac{n-1}{2n},\frac{n+1}{2n}}$,
- the twisted Dirac operator : $g^{ij_1}(\mathrm{vol}_g)_{j_1\ldots j_n}\gamma^{j_2}\ldots\gamma^{j_n}\nabla_i \in \mathcal{D}^{\frac{n-1}{2n},\frac{n+1}{2n}}$,
- the operators : $\mathcal{N}(\Delta R^s) \in \mathrm{D}^{\frac{n-2s-1}{2n},\frac{n+2s+1}{2n}}$, of order $2s+1$.

Conformally equivariant quantization of the supercotangent bundle

Theorem

There exists (generically) a unique quantization $\mathcal{Q}^{\lambda,\mu} : \mathcal{S}^\delta[\xi] \to \mathrm{D}^{\lambda,\mu}$ *which is conformally equivariant, i.e. such that* $\mathcal{L}_X^{\lambda,\mu}\mathcal{Q}^{\lambda,\mu} = \mathcal{Q}^{\lambda,\mu}L_X^\delta$ *for all* $X \in \mathrm{o}(p+1, q+1)$.

Remark : the conformal invariants correspond to each other via

$$\mathcal{S}^\delta \xrightarrow{\mathcal{Q}^{\lambda,\mu}} \mathrm{D}^{\lambda,\mu},$$

as soon as $\mathcal{Q}^{\lambda,\mu}$ exists.

Thanks !

Reference : *Quantification conformément équivariante des fibrés supercotangents*, Jean-Philippe Michel, thèse, tel-00425576 version 1.