On conformally equivariant quantization

Jean-Philippe Michel

Unité de Recherche en Mathématiques,
Université du Luxembourg

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Quantization : "simplest case"

	classical	quantum
Phase space	$T^*\mathbb{R}^n$	$\mathrm{L}^2(\mathbb{R}^n)$
Observables	$\mathrm{Pol}(T^*\mathbb{R}^n)$ graded algebra	$\mathcal{D}(\mathbb{R}^n)$ filtered algebra.

The two algebras are linked by : $\mathrm{Pol}(T^*\mathbb{R}^n) \simeq \bigoplus_{k=0}^{\infty} \mathcal{D}_k(\mathbb{R}^n)/\mathcal{D}_{k-1}(\mathbb{R}^n)$.

$$\text{quantization} = (\text{symbol map})^{-1}$$

Examples

- The normal ordering : $\mathcal{N} : P^{i_1\cdots i_k}(x)p_{i_1}\cdots p_{i_k} \mapsto P^{i_1\cdots i_k}(x)\partial_{i_1}\cdots\partial_{i_k}$.
- Any map of the form : $\mathcal{N} \circ (\mathrm{Id} + \text{Operator lowering the degree})$.
- The Weyl quantization : $\mathcal{W} = \mathcal{N} \circ \exp(\frac{D}{2})$, where $D = \partial_i \partial_{p_i}$.

Problem : how to fix a preferred one ?

Idea : fix the quantization by symmetries of the configuration space.

Equivariant quantization [Duval, Lecomte, Ovsienko '99]

First Definition Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$. The quantization $\mathcal{Q}$ is $\mathfrak{g}$-equivariant if the following diagramm is commutative for every vector field $X \in \mathfrak{g}$,

$$\begin{array}{ccc} \mathrm{Pol}(T^*\mathbb{R}^n) & \xrightarrow{\mathcal{Q}} & \mathcal{D}(\mathbb{R}^n) \\ \uparrow L_X & & \uparrow \mathcal{L}_X \\ \mathrm{Pol}(T^*\mathbb{R}^n) & \xrightarrow{\mathcal{Q}} & \mathcal{D}(\mathbb{R}^n) \end{array} \qquad (1)$$

where L_X and $\mathcal{L}_X$ denote the action of X on $\mathrm{Pol}(T^*\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$.

Facts

- There exists no $\mathrm{Vect}(\mathbb{R}^n)$-equivariant quantization.
- The normal ordering is an $\mathrm{aff}(\mathbb{R}^n)$-equivariant quantization.

The $\mathrm{Vect}(\mathbb{R}^n)$*-module structures at play* :

- $\mathcal{F}^\lambda = (\mathcal{C}^\infty(\mathbb{R}^n), \ell_X^\lambda)$ is the module of λ-weighted tensor densities, with $\ell_X^\lambda = X + \lambda \mathrm{Div}(X)$. Geometrically $\mathcal{F}^\lambda \simeq \Gamma(|\Lambda T^*\mathbb{R}^n|^{\otimes\lambda})$.
- $\mathcal{D}^{\lambda,\mu} = (\mathcal{D}(\mathbb{R}^n), \mathcal{L}_X^{\lambda,\mu})$ is the module of differential operators $A : \mathcal{F}^\lambda \to \mathcal{F}^\mu$, with the adjoint action $\mathcal{L}_X^{\lambda,\mu} A = \ell_X^\mu A - A\ell_X^\lambda$.
- $\mathcal{S}^\delta = (\mathrm{Pol}(T^*\mathbb{R}^n), L_X^\delta)$ is the induced graded module of symbols, with the natural action $L_X^\delta = X - p_j(\partial_i X^j)\partial_{p_i} + \delta \mathrm{Div}(X)$.

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Definition

A $\mathfrak{g}$*-equivariant quantization is a isomorphism of* $\mathfrak{g}$*-modules* $\mathcal{Q}^{\lambda,\mu} : \mathcal{S}^\delta \to \mathcal{D}^{\lambda,\mu}$*, whose inverse is a symbol map.*

The conformal case

Setting : $(\mathbb{R}^n, [\eta])$ and $\mathfrak{cf} \simeq \mathrm{o}(p+1, q+1)$.

Theorem (DLO '99)

Let $\lambda, \mu \in \mathbb{R}$, and $\delta = \mu - \lambda$ be the shift. For every $k \in \mathbb{N}$, there exists a unique $\mathfrak{cf}$-equivariant quantization $\mathcal{Q}^{\lambda,\mu} : \mathcal{S}^\delta_k \to \mathcal{D}^{\lambda,\mu}_k$ iff $\delta \notin I_k$, a finite subset of positive rationnal numbers called resonances.

Restricting to $\mathcal{S}^\delta_2$, the resonances δ are

δ	$\frac{2}{n}$	$\frac{n+2}{2n}$	1	$\frac{n+1}{n}$	$\frac{n+2}{n}$
λ	$\frac{n-2}{2n}$	$0, \frac{n-2}{2n}$	0	$0, -\frac{1}{n}$	$-\frac{1}{n}$
μ	$\frac{n+2}{2n}$	$\frac{n+2}{2n}, 1$	1	$\frac{n+1}{n}, 1$	$\frac{n+1}{n}$

(2)

The corresponding modules $\mathcal{D}^{\lambda,\mu}_2$ are exceptionnal.

Problematic

Determination and interpretation of the sets I_k of resonances

1. DLO ('99) provide subsets of $\mathbb{Q}_+^*$ which are not optimal, as shown by the explicit CEQ of $\mathcal{S}_3^\delta$ by Loubon-Djounga ('01).
2. The projective case is fully understood in terms of cohomology of $\mathrm{sl}(n+1,\mathbb{R})$-modules [Lecomte, Ovsienko '00].
3. Silhan ('09) conjectures the minimality of the sets of resonances that he obtains, and remarks that they corresponds to the existence of some conformal invariant operators on $\mathcal{S}^\delta$.

Theorem (I)

Let $F^\delta = (F, L_X^\delta)$ be a $\mathfrak{cf}$-submodule of $\mathcal{S}_k^\delta$ for any $\delta \in \mathbb{R}$. The $\mathfrak{cf}$-equivariant quantization exists and is unique on F^δ iff there exists no $\mathfrak{cf}$-invariant operator from F^δ to $\mathcal{S}_{k-l}^\delta$, for some $l = 1, \dots, k$.

Comments : we have to prove that, non-existence of $\mathrm{CEQ} \Rightarrow$ $\mathfrak{cf}$-invariant operator.

Ingredients of the proof : interpretation of the CEQ in terms of cohomology of $\mathfrak{cf}$-modules.

Cohomology of $\mathfrak{g}$-modules :

Let (M, L^M) be a $\mathfrak{g}$-module. The k-cochains are maps $\Lambda^k\mathfrak{g} \to M$, and

$$
\begin{aligned}
d\phi(X) &= L^M_X \phi \\
d\gamma(X, Y) &= L^M_X \cdot \gamma(Y) - L^M_Y \cdot \gamma(X) - \gamma([X, Y]).
\end{aligned}
$$

Consequently, $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$ and $\{$1-coboundaries$\} \simeq M/M^{\mathfrak{g}}$.

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Splitting of exact sequences :
The exact sequence of $\mathfrak{g}$-modules, with linear section τ,

$$0 \longrightarrow (A, L^A) \xrightarrow{\ \iota\ } (B, L^B) \underset{\tau}{\overset{\sigma}{\rightleftarrows}} (C, L^C) \longrightarrow 0, \qquad (3)$$

defines a 1-cocycle $\gamma = \iota^{-1}(L^B - \tau \circ L^C)$, with values in $\mathrm{Hom}(C, A)$.

Lemma

The above sequence is split iff $\gamma = d\phi$ is a coboundary and $\tau + \phi$ is the sought section.

Cohomological interpretation of $\mathfrak{g}$-EQ

The $\mathfrak{g}$-EQ of $\mathcal{S}_k^\delta$ is a section of the exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow \mathcal{D}_{k-1}^{\lambda,\mu} \longrightarrow \mathcal{D}_k^{\lambda,\mu} \longrightarrow \mathcal{S}_k^\delta \longrightarrow 0. \qquad (4)$$

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The $\mathfrak{g}$-EQ of $F^\delta \subset \mathcal{S}_k^\delta$ is given by ϕ_k^δ, which exists iff

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}_{k-l}^\delta & \longrightarrow & \mathcal{D}_k^{\lambda,\mu}/\mathcal{D}_{k-l-1}^{\lambda,\mu} & \longrightarrow & \mathcal{D}_k^{\lambda,\mu}/\mathcal{D}_{k-l}^{\lambda,\mu} & \longrightarrow & 0 \\
 & & & & & {\scriptstyle \exists\phi_l^\delta}\nwarrow & \uparrow{\scriptstyle \phi_{l-1}^\delta} & & \\
 & & & & & & F^\delta & &
\end{array} \tag{5}$$

for succesively all $l = 1, \ldots, k$. The cocycles $\gamma_l^\delta = (\mathcal{L}^{\lambda,\mu} - L^\delta) \circ \phi_{l-1}^\delta$ must be coboundaries $\gamma_l^\delta = d\psi_l^\delta$, and define then $\phi_l^\delta = \phi_{l-1}^\delta + \psi_l^\delta$.

The conformal case

We have $\mathfrak{cf} = \mathfrak{ce} + [\mathfrak{ce}, X_i]$ with $\mathfrak{ce}$ the Lie algebra of affine conformal transformations and X_i an inversion.

Consequence : CEQ exists on F^δ iff it exists $\psi_l^\delta \in \mathrm{Hom}_{\mathfrak{ce}}(F^\delta, \mathcal{S}_{k-l}^\delta)$ such that

$$\gamma_l^\delta(X_i) = [L_{X_i}^\delta, \psi_l^\delta].$$

Fact : there exists a finite dimensionnal space E independent of δ, which contains $\gamma_l^\delta(X_i)$ and the image of the linear map

$$[L_{X_i}^\delta, \cdot] : \mathrm{Hom}_{\mathfrak{ce}}(F^\delta, \mathcal{S}_{k-l}^\delta) \to E,$$

for every δ and λ. Thus, CEQ exists on F^δ iff $\gamma_l^\delta(X_i) \in \mathrm{im}([L_{X_i}^\delta, \cdot])$.

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Proof of Theorem I : Let δ_0 be a resonance.
Generically $\gamma_l^\delta(X_i) \in \mathrm{im}([L_{X_i}^\delta, \cdot])$, so $\mathrm{rk}([L_{X_i}^{\delta_0}, \cdot]) < \mathrm{rk}([L_{X_i}^\delta, \cdot])$. The rank Thm implies the existence of a $\mathfrak{cf}$-invariant operator from F^{δ_0} to $\mathcal{S}_{k-l}^{\delta_0}$.

Aim : classification of $\mathfrak{cf}$-invariant operators on $\mathcal{S}^\delta$

Fact : an operator $A : \mathcal{S}_k^\delta \to \mathcal{S}^\delta$ invariant by translations and dilations is a differential operator [Lecomte, Ovsienko '99].

Proposition (Weyl-Brauer)

The algebra of isometric invariant differential operators on $\mathcal{S}^\delta$ is generated by

$$\begin{aligned} R &= \eta^{ij} p_i p_j, \qquad \mathcal{E} = p_i \partial_{p_i}, \qquad T = \eta_{ij} \partial_{p_i} \partial_{p_j}, \\ G &= \eta^{ij} p_i \partial_i, \qquad D = \partial_i \partial_{p_i}, \qquad L = \eta^{ij} \partial_i \partial_j. \end{aligned}$$

They are all $\mathfrak{ce}$-invariant operators but from $\mathcal{S}^\delta$ to $\mathcal{S}^{\delta'}$ following

values of $n(\delta' - \delta)$	-2	0	2
$\mathfrak{ce}$-invariant operators	T	$\mathcal{E}, D$	R, G, L

(6)

Restriction to $\mathfrak{cf}$-submodules

The $\mathfrak{cf}$-invariant operators $\mathcal{E}$ and RT give rise to the decomposition

$$\mathcal{S}^\delta = \bigoplus_{k,s\in\mathbb{N},\, 2s\leq k} \mathcal{S}^\delta_{k,s},$$

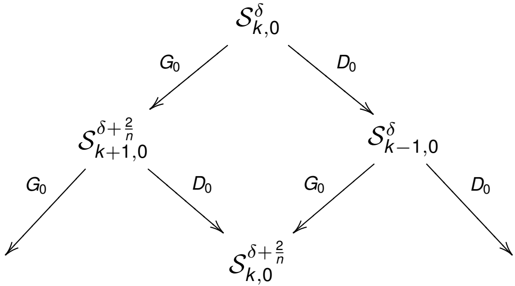
where $\mathcal{S}^\delta_{k,s}$ is the space of homogeneous symbols of degree k of the form $R^s Q$ with $TQ = 0$. The following commutative diagramm

$$\begin{array}{ccc} \mathcal{S}^\delta_{k,s} & \longrightarrow & \mathcal{S}^\delta_{l,t} \\ {\scriptstyle T^s}\downarrow & & \uparrow{\scriptstyle R^l} \\ \mathcal{S}^{\delta-\frac{2s}{n}}_{k-2s,0} & \longrightarrow & \mathcal{S}^{\delta-\frac{2t}{n}}_{l-2t,0}. \end{array} \qquad (7)$$

proves that it suffices to work with G_0, D_0, L_0 the restrictions and corestrictions to $\ker T$ of G, D, L. Unique relation $[D_0, G_0] = L_0$.

Proposition

$$\mathrm{Hom}_{\mathfrak{ce}}(\mathcal{S}^{\delta}_{k,0},\mathcal{S}^{\delta'}_{k',0}) = \begin{cases} 0 \quad \text{if } j=\frac{n}{2}(\delta'-\delta)-\max(k'-k,0)\notin\mathbb{N}, \text{ else} \\ (G_0)^{k'-k}\left\langle L_0^j, G_0L_0^{j-1}D_0,\ldots,G_0^jD_0^j\right\rangle \quad \text{for } l-k\leq 0 \\ \left\langle L_0^j, G_0L_0^{j-1}D_0,\ldots,G_0^jD_0^j\right\rangle (D_0)^{k-k'} \quad \text{for } k-l>0. \end{cases}$$



(8)

Classification of conformally invariant operators on $\mathcal{S}^\delta$

Theorem (II)

$\mathrm{Hom}_{\mathfrak{cf}}(\mathcal{S}^\delta_{k,s}, \mathcal{S}^\delta_{k',s'})$ *is either trivial or of dimension* 1*, generated by*

- $R^{s'} G_0^g T^s$ *if* $k - k' = s - s' = g$ *and* $\delta = \frac{2s+1-g}{n}$,
- D^d *if* $k - k' = d$, $s = s'$ *and* $\delta = 1 + \frac{2k-d-1}{n}$,
- $R^{s'} \mathcal{L}_0^l T^s$ *if* $k - k' = 2s - 2s' = 2l$ *and* $\delta = \frac{1}{2} + \frac{k-l}{n}$,

where the operator $\mathcal{L}_0^l$ *is of the form* $L_0^l + a_1 G_0 L_0^{l-1} D_0 + \ldots + a_j G_0^j D_0^j$ *for* $a_i \in \mathbb{R}$.

This is a particular case of general classification [Baston, Eastwood, Rice '87/90].

Consequence : $\mathcal{S}^\delta_{k,s}$ is indecomposable as $\mathfrak{cf}$-module. Irreducible generically ?

Main result

Theorem (III)

The CEQ *exists and is unique on* $\mathcal{S}^{\delta}_{k,s}$ *iff* $\delta \notin I_{k,s} = I^G_{k,s} \cup I^L_{k,s} \amalg I^D_{k,s}$*, where*

$$
\begin{aligned}
I^G_{k,s} &= \left\{ \frac{2s+1-g}{n} \mid g = 1, \dots, s \right\}, \\
I^L_{k,s} &= \left\{ \frac{1}{2} + \frac{k-l}{n} \mid l = 1, \dots, s \right\}, \\
I^D_{k,s} &= \left\{ 1 + \frac{2k-d-1}{n} \mid d = 1, \dots, k \right\},
\end{aligned}
\tag{9}
$$

are the sets of shifts δ *for which operators of the form* $R^{s'} G^g_0 T^s$*,* $R^{s'} \mathcal{L}^l_0 T^s$ *and* D^d *are conformally invariant on some* $\mathcal{S}^{\delta}_{k,s}$*.*

Theorem (IV)

Let $\delta \in I_{k,s}$. Restricted to the submodule $\mathcal{S}^{\delta}_{k,s}$, the CEQ $\mathcal{Q}^{\lambda,\mu} : \mathcal{S}^{\delta}_{k,s} \to \mathcal{D}^{\lambda,\mu}$ *exists iff*

$$\lambda = \begin{cases} \frac{n+2(g'-s-1)}{2n},\ g' = 1, \ldots, g & \text{if } \delta = \frac{2s+1-g}{n} \in I^G_{k,s} \setminus I^L_{k,s}, \\ -\frac{k-d'-\chi_{s\neq 0}}{n},\ d' = 1, \ldots, d & \text{if } \delta = 1 + \frac{2k-d-1}{n} \in I^D_{k,s}, \\ \frac{n-2s}{2n} \text{ or } -\frac{k-l-1}{n}, & \text{if } \delta = \frac{1}{2} + \frac{k-l}{n} \in I^L_{k,s} \setminus I^G_{k,s}, \\ \frac{n-2s}{2n}, & \text{if } \delta = \frac{1}{2} + \frac{k-l}{n} \in I^L_{k,s} \cap I^G_{k,s}. \end{cases} \tag{10}$$

Interpretation of the associated exceptionnal $\mathcal{D}^{\lambda,\mu}$?

Definition of higher symmetries of Δ^k

Let Δ^k be the k^{th} power of the Laplacian $\Delta = \eta^{ij}\partial_i\partial_j$ of $(\mathbb{R}^n, \eta)$.
It is covariant under $X \in \mathrm{Vect}(\mathbb{R}^n)$ if : $[X, \Delta^k] = f\Delta^k$ for $f \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Definition

A higher symmetry of Δ^k is a differential operator D such that $[D, \Delta^k] = A\Delta^k$ for some $A \in \mathcal{D}(\mathbb{R}^n)$.

Equivalently it is a pair (D_1, D_2) such that $\Delta^k D_1 = D_2 \Delta^k$.

Example : the HS of 1^{st} order for Δ^k are the pairs $(\ell_X^\lambda, \ell_X^\mu)$, with $\lambda = \frac{n-2k}{2n}$, $\mu = \frac{n+2k}{2n}$ and $X \in \mathfrak{cf}$. So, $\Delta^k \in \mathcal{D}^{\lambda,\mu}$ is conformally invariant.

Aim : Determine the higher symmetries of Δ^k modulo the trivial ones $(A\Delta^k)$, i.e. the kernel of the $\mathfrak{cf}$-invariant operator

$$\mathrm{HSQ} : \begin{array}{rcl} \mathcal{D}^{\lambda,\lambda}/(\Delta^k) & \to & \mathcal{D}^{\lambda,\mu}/(\Delta^k) \\ D & \mapsto & \Delta^k D \end{array}$$

The determination of the higher symmetries of Δ^k :

- $k = 1$ by Eastwood in ’02,
- $k = 2$ by Eastwood, Leistner in ’06,
- general case by Gover, Silhan in ’09.

Correspondence of classical and quantum HS

Idea : "quantum" HS correspond to classical ones via the CEQ.

	classical	quantum
Hamiltonian	$R = \eta^{ij} p_i p_j$	$\Delta = \eta^{ij} \partial_i \partial_j$
Symmetries	P s.t. $\{R, P\} = QR$	D s.t. $[\Delta, D] = A\Delta$
	$P \in \ker G_0$	$D \in \ker \mathrm{HSQ}$

$\ker G_0$ is the space of conformal Killing tensors.

Lemma

$$\begin{array}{ccc}
\mathcal{D}^{\lambda,\lambda}/(\Delta^k) & \xrightarrow{\mathrm{HSQ}} & \mathcal{D}^{\lambda,\mu}/(\Delta^k) \\
\uparrow{\scriptstyle \mathcal{Q}^{\lambda,\lambda}} & & \uparrow{\scriptstyle \mathcal{Q}^{\lambda,\mu}} \\
\bigoplus_{l=0}^{k-1} \mathcal{S}^0_{*,l} & \xrightarrow[\mathrm{HSC}]{} & \bigoplus_{l=0}^{k-1} \mathcal{S}^{\frac{2k}{n}}_{*,l}
\end{array} \tag{11}$$

is commutative with HSC *proportionnal to* $R^{k-l-1} G_0^{2l+1} T^l$ *on* $\mathcal{S}^0_{*,l}$.

Theorem

The HS *of* Δ^k *are the* $\mathcal{Q}^{\lambda,\lambda}(P)$ *for* P *a generalized conformal Killing tensor, i.e. in the kernel of* $R^{k-l-1}G_0^{2l+1}T^l$*. Moreover, we have for such a* P*,*

$$\Delta^k \mathcal{Q}^{\lambda,\lambda}(P) = \mathcal{Q}^{\mu,\mu}(P)\Delta^k.$$

Idea :

$\mathcal{Q}^{\mu,\mu} \circ (\mathcal{Q}^{\lambda,\lambda})^{-1} : \mathcal{D}^{\lambda,\lambda} \to \mathcal{D}^{\mu,\mu}$ is the unique $\mathfrak{cf}$-invariant operator preserving the principal symbol.

Thanks !

Reference :

Conformally equivariant quantization and higher symmetries of conformal powers of the Laplacian, Jean-Philippe Michel, in preparation.