From symplectic to spin geometry

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Quantization for spinless system

classical	quantum	link
symplectic mfld (T^*M, ω)	(pre-)Hilbert space $\mathcal{F}_c^{\frac{1}{2}} = \Gamma_c(\lvert\Lambda^n T^*M\rvert^{\otimes\frac{1}{2}})$	vertical polarization
co. graded P. alg. $\mathrm{Pol}(T^*M)$	ass. filtered alg. $\mathcal{D}^{\frac{1}{2},\frac{1}{2}}(M)$	$\mathrm{Pol}_k \simeq \mathcal{D}_k^{\frac{1}{2},\frac{1}{2}}/\mathcal{D}_{k-1}^{\frac{1}{2},\frac{1}{2}}$ $\sigma_{p+q-1}([P,Q]) = \{\sigma_p(P), \sigma_q(Q)\}$
preserve $\{\cdot,\cdot\}$ $\mathrm{Vect}(M) \hookrightarrow \mathrm{Pol}_1$	preserve $[\cdot,\cdot]$ $\mathrm{Vect}(M) \hookrightarrow \mathcal{D}_1^{\frac{1}{2},\frac{1}{2}}$	$\mathrm{GQ}(J_X) = X^i\partial_i + \frac{1}{2}(\partial_i X^i)$.

Quantization for spin system

Let (M, g) be a pseudo-Riemannian spin manifold with $\dim M = 2n$.
Remind that $\mathbb{C}\mathrm{l}(V, g) = \bigotimes V / \langle u \otimes v + v \otimes u + 2g(u, v) \rangle$.

	$\Gamma(\mathrm{S}) \otimes \mathcal{F}_c^{\frac{1}{2}}$
$\mathrm{Pol}(T^*M) \otimes \Gamma(\mathbb{C}\mathrm{l}(M, g))$	$\mathrm{D}^{\frac{1}{2},\frac{1}{2}} \ni P^{i_1 \cdots i_k}_{j_1 \cdots j_\kappa}(x)\gamma^{j_1} \ldots \gamma^{j_\kappa}\partial_{i_1} \ldots \partial_{i_k}$

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Questions :

- symplectic form on $\mathcal{M}$?
- Polarization and construction of S ?
- Quantum action of vector fields ?
- Poisson bracket from the commutator, which graduation ?

1. From symplectic spinors to spinors
   - Algebraic point of view
   - Geometrization
2. Actions of vector fields
   - On the supercotangent bundle $\mathcal{M}$
   - On the spinor bundle S
3. D as quantization of *Sb*

Quantization of symplectic even and odd vector spaces

Let V be a vector space, with coordinates (ξ^i), $\dim V = 2n$. Remind that $\mathcal{W}(V,\omega) = \bigotimes V / \langle u \otimes v - v \otimes u - \omega(u,v) \rangle$.

	Even (V,ω)	odd $(\Pi V, g)$
graded alg.	$S(V^*)$	ΛV^*
symplectic form	$\omega_{ij} d\xi^i \wedge d\xi^j$	$g_{ij} d\xi^i \wedge d\xi^j$
symmetries	$(S^2(V^*), \{\cdot,\cdot\}) \simeq \mathrm{sp}(V,\omega)$	$(\Lambda^2 V^*, \{\cdot,\cdot\}) \simeq \mathrm{o}(V,g)$
Moyal $*$-product	$(S(V^*), *) \simeq \mathcal{W}(V^*, \omega^{-1})$	$(\Lambda V^*, *) \simeq \mathbb{C}\mathrm{l}(V^*, g^{-1})$
Group action	$\mathrm{Mp}(V,\omega) = \exp(\mathrm{sp}(V,\omega))$	$\mathrm{Spin}(V,g) = \exp(\mathrm{o}(V,g))$

(Voronov '90)

	Even (V,ω)	odd $(\Pi V, g)$
Darboux coord.	$\omega = dp_i \wedge dx^i$	$g = \varepsilon_i d\xi^i \wedge d\xi^i \quad ; \varepsilon_i = \pm 1$ Riemannian : $\varepsilon_i = 1$
Lagrangian	$V = \langle x^i \rangle \oplus \langle p_i \rangle$	$V \otimes \mathbb{C} = P \oplus \bar{P}$ $P = \left(\frac{1+iJ}{\sqrt{2}}\right) V$, $J^2 = -1$ Herm. str.
Polarized fct.	$S(\langle x^i \rangle)$	ΛP^*
Quant. fct.	$S(\langle x^i \rangle)(1 + \langle p_i \rangle)$	$\Lambda P^* (1 + \langle \bar{P}^* \rangle)$
Rep. space	$\mathcal{H} = S(\langle x^i \rangle) \otimes \det^{\frac{1}{2}}$ $\mathrm{End}(\mathcal{H}) \simeq \mathcal{W}(V^*, \omega^{-1})$	$\mathcal{H} = \Lambda P^* \otimes \mathrm{Ber}^{\frac{1}{2}}$ $\mathrm{End}(\mathcal{H}) \simeq \mathbb{C}\mathrm{l}(V^*, g^{-1})$

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	Even (V, ω)	odd $(\Pi V, g)$
Darboux coord.	$\omega = dp_i \wedge dx^i$	$g = \varepsilon_i d\xi^i \wedge d\xi^i \times \frac{\hbar}{2i}$; $\varepsilon_i = \pm 1$ Riemannian : $\varepsilon_i = 1$
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Symplectic supermanifold

Supermanifold := loc. ringed space $\mathcal{N} = (N, \mathcal{O})$ s.t. $\mathcal{O}_{|U} \simeq \mathcal{C}^\infty(\mathbb{R}^n) \otimes \Lambda\mathbb{R}^m$.

- Batchelor Theorem ('79) : $\mathcal{N} \longleftrightarrow (N, E)$, $\Pi E = (N, \Gamma(\Lambda E^*))$.
  Example : $\mathcal{M} = T^*M \oplus \Pi TM$ supercotangent bundle over (M, g).
- Rothstein Theorem ('91) : $(\mathcal{N}, \omega) \longleftrightarrow (N, \omega_0, E, g, \nabla)$.
  $\Rightarrow (\mathcal{M}, \omega)$ over (M, g), given by $\omega = d\alpha$,

$$\alpha = p_i dx^i + \frac{\hbar}{2\mathrm{i}} g_{ij} \xi^i d^\nabla \xi^j.$$

Darboux coordinates (non-tensorial !) :

$$\tilde{\xi}^a = \theta^a_i \xi^i \quad \text{and} \quad \tilde{p}_i = p_i + \frac{\hbar}{2\mathrm{i}} \omega_{iab} \tilde{\xi}^a \tilde{\xi}^b,$$

with $\theta^a_i \theta^b_j g^{ij} = \eta^{ab}$ and $\omega_{iab} = \partial_{\tilde{\xi}^a} \nabla_i \tilde{\xi}_b$.

The momentum of a rotation is :

$$\mathcal{J}_{X_{ij}} = p_i x_j - x_j p_i + \frac{\hbar}{\mathrm{i}} \xi_i \xi_j,$$

hence the spin is $S_{ij} = \frac{\hbar}{\mathrm{i}} \xi_i \xi_j$, generating $(\Omega^2_x(M), \{\cdot, \cdot\}) \simeq \mathrm{o}(p, q)$.

Example

Rotating particle e.o.m.$=$ Papapetrou equations $=$ Hamiltonian flows of $g^{ij} p_i p_j$:

$$\dot{x}^j \nabla_j \dot{x}^i = -\frac{1}{2} g^{ik} R(S)_{jk} \dot{x}^j,$$
$$\dot{x}^k \nabla_k S^{ij} = 0.$$

GQ of the supercotangent bundle

- Locally : merging of the GQ of $T^*\mathbb{R}^n$ and $\Pi\mathbb{R}^n$.
  Explicit formulae in Darboux coord.
- Globally : we need a polarization in the fibers $\Pi T_x M$,
  $\hookrightarrow$ almost Hermitian structure on (M, g).
- Polarized functions : $\Lambda(T^{1,0\,*}M)$, holomorphic diff. forms.
  $\Rightarrow \Gamma(\mathbb{C}\mathrm{l}(M, g)) \simeq \mathrm{End}(\Lambda(T^{1,0\,*}M) \otimes \mathrm{Ber}^{\frac{1}{2}})$, spinor bundle S !

Remark : coincide with usual construction of S over pseudo-Hermitian manifold :

- almost Hermitian structure $\Rightarrow \mathrm{Spin}^c$-structure,
- $+$ existence $\mathrm{Ber}^{\frac{1}{2}} (= K^{\frac{1}{2}}) \Rightarrow \mathrm{Spin}$-structure,
- induced action of $\mathrm{u}(n)$ on S is the natural one on $\Lambda(T^{1,0\,*}M) \otimes \mathrm{Ber}^{\frac{1}{2}}$.

$\mathrm{Vect}(M)$ action on the supercotangent bundle

- Tensorial action : $\mathbb{L}_X = X^i\partial_i - p_j(\partial_i X^j)\partial_{p_i} + \xi^j(\partial_j X^i)\partial_{\xi^i}$.
- Hamiltonian action : $L_{\tilde{X}}\alpha = 0$, given by ?

Lemma

$$\tilde{X} = X^i\partial_i^\nabla + Y_{ij}\xi^j\partial_{\xi_i} - p_j\nabla_i X^j\partial_{p_i} + \frac{\hbar}{2\mathrm{i}}\left(R^k_{lij}\xi_k\xi^l X^j - (\nabla_i Y_{kl})\xi^k\xi^l\right)\partial_{p_i},$$

where Y is an arbitrary 2-form on M, depending linearly on X.

1. $Y = 0 \Rightarrow \mathrm{Vect}(M)$-action iff ∇ is flat.
   Lift by ∇ of Ham. action on T^*M, "moment map" : $J_X = p_i X^i$.
2. $L_{\tilde{X}}\beta = f\beta$, where $\beta = g_{ij}\xi^i dx^j$, $d\beta$ odd sympl. form on ΠTM.
   Then, $\mathrm{conf}(M, g)$-action, $Y_{ij} = -\partial_{[i}X_{j]}$.
   Moment map : $\mathcal{J}_X = p_i X^i + \frac{\hbar}{2\mathrm{i}}\xi^j\xi^k\partial_{[k}X_{j]}$.

Vector fields action on the spinor bundle

We expect that an action of X on $\Gamma(S)$ is of the form

$$\nabla_X + Y_{ij}\gamma^i\gamma^j,$$

with Y as above. For the representation space $\Gamma(S)$, GQ is a Lie algebra morphism on : $Sb_{0,0} \oplus Sb_{1,0} \oplus Sb_{0,1} \oplus Sb_{0,2}$, where $Sb_{k,\kappa} = \mathrm{Pol}_k(T^*M) \otimes \Omega^\kappa_{\mathbb{C}}(M)$. Hence,

$$\text{Hamiltonian lift} \quad \Longleftrightarrow \quad \text{Lie derivative of spinors .}$$

Theorem

- $\mathrm{GQ}(J_X) = \nabla_X$, *spinor covariant derivative ;*
- $\mathrm{GQ}(\mathcal{J}_X) = \ell_X = \nabla_X + \frac{1}{4}\partial_{[k}X_{j]}\gamma^j\gamma^k$, *spinor Lie derivative (Kosmann'72).*

Algebra of spinor differential operators D as a deformation of *Sb*

Spinless case : $\sigma_{p+q-1}([P,Q]) = \{\sigma_p(P), \sigma_q(Q)\}$.

Dirac operator : $\gamma^i \nabla_i$,
Lichnerowicz Laplacian : $[\gamma^i \nabla_i, \gamma^i \nabla_i] = \Delta + \frac{R}{4}$, second order diff. op.
$\Rightarrow$ usual filtration by order $\mathrm{D}_0 \subset \mathrm{D}_1 \subset \cdots$ does not fit.

Idea : *Hamiltonian* filtration with ∇_i of order 1 and γ^i of order $\frac{1}{2}$.

$$[\nabla_i, \nabla_j] = R_{ijkl}\gamma^k\gamma^l \quad \text{and} \quad [\gamma^i, \gamma^j] = -2g^{ij}.$$

Theorem

D *admits a filtration s.t.* $\mathrm{D}_K/\mathrm{D}_{K-\frac{1}{2}} \simeq Sb_K = \bigoplus_{2k+\kappa=K} Sb_{k,\kappa}$ *and* $(Sb, \{\cdot,\cdot\})$ *is the graded version of* $(\mathrm{D}, [\cdot,\cdot])$

$$\sigma_{K+L-1}([P,Q]) = \{\sigma_K(P), \sigma_L(Q)\}.$$

D as deformation of the $\mathrm{conf}(M, g)$-module Sb

Let us suppose that (M, g) is conformally flat, loc. $g_{ij} = F\eta_{ij}$.

- D, adjoint action $\mathcal{L}_X A = [\ell_X, A]$, preserves Ham. and usual filtration.
- Sb, Ham. action preserves Ham. graduation, on $Sb_{*,\kappa}$:

$$\tilde{X} = \mathbb{L}_X - \frac{\kappa}{n}(\nabla_i X^i) + \left(R^k_{lij}\xi_k \xi^l X^j - (\nabla_i Y_{kl})\xi^k \xi^l \right) \partial_{p_i}.$$

As $\sigma(\mathrm{GQ}(\mathcal{J}_X)) = \mathcal{J}_X$, we have $\sigma([L_x, \cdot]) = \{\mathcal{J}_X, \cdot\}$.
$\Rightarrow$ (Sb, L) is the graded version of $(\mathrm{D}, \mathcal{L})$.

Proposition

$\mathcal{T} = \bigoplus_\kappa Sb_{*,\kappa} \otimes \mathcal{F}^{-\frac{\kappa}{n}}$ endowed with the tensorial action $\mathbb{L}^* = \mathbb{L}^{-\frac{\kappa}{n}}$ on $Sb_{*,\kappa}$ is the graded version of (Sb, L) and $(\mathrm{D}, \mathcal{L})$ for the usual filtration.

Main result

Theorem

There exists a unique morphism of $\mathrm{conf}(M, g)$*-modules preserving the principal symbol for the usual filtration*

$$\mathrm{S}^{\delta}_{\mathcal{T}} : \mathcal{T}^{\delta} \to \mathcal{S}b^{\delta},$$

it is named the conformally equivariant superization.

Theorem

There exists a unique morphism of $\mathrm{conf}(M, g)$*-modules preserving the principal symbol for the Ham. filtration,*

$$\mathcal{Q}^{\lambda,\mu} : \mathcal{S}b^{\delta} \to \mathrm{D}^{\lambda,\mu},$$

if $\mu - \lambda = \delta$*. It is named the conformally equivariant quantization.*

Examples and applications

Examples :

1. $\mathrm{S}^0_{\mathcal{T}}(J_X = p_i X^i) = \mathcal{J}_X = p_i X^i + \frac{\hbar}{2\mathrm{i}} \xi^j \xi^k \partial_{[k} X_{j]}$.
2. Dirac operator from De Rham differential : $\mathcal{Q} \circ \mathrm{S}_{\mathcal{T}}(p_i \xi^i) = \gamma^i \nabla_i$.

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Hamiltonian description of the Killing-Yano tensors, i.e. diff. κ-forms T s.t. $\nabla_{[i_0}T_{i_1\cdots i_\kappa]} = 0$,

$$T \in \mathcal{T}^{-\frac{1}{n}}_{0,\kappa} \text{ s.t. } \{p_i\xi^i, \mathrm{S}^0_{\mathcal{T}}(\phi(T))\} = 0 \;\Leftrightarrow\; T \text{ is a Killing-Yano tensor.}$$

Thanks !

References :

Quantification conformément équivariante des fibrés supercotangents, Jean-Philippe Michel, thèse, tel-00425576 version 1.

Conformal geometry of the supercotangent and spinor bundles, Jean-Philippe Michel, arXiv :1004.1595v1.

Conformally equivariant quantization of supercotangent bundles, Jean-Philippe Michel, in preparation.