Higher symmetries of Yamabe Laplacian

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Problematic

Let (M, g) be a pseudo-Riemannian manifold, $c \in \mathbb{R}$ and $\Delta_c = \nabla_i g^{ij} \nabla_j + cR$ a Laplacian operator on M.

Find the differential operators $D_1 \in \mathcal{D}_2(M)$ such that

1. $[\Delta_c, D_1] = 0$,
   symmetries of Δ_c, preserve eigenspaces,
2. $\Delta_c D_1 = D_2 \Delta_c$ for some $D_2 \in \mathcal{D}_2(M)$,
   conformal symmetries of Δ_c, preserve $\ker \Delta_c$.

1. Reduction of the problem
   - First order symmetries
   - Symmetries of (null) geodesic flow
2. Known results
3. A new approach
   - Conformally invariant quantization
4. Examples
   - Symmetries
   - Obstruction to symmetries

First order (conformal) symmetries of Δ_c

The zero order (conformal) symmetries are the constants. Up to constants, first order (conformal) symmetries are given by

1. $[\Delta_c, X] = 0$ iff X is a Killing vector field, i.e. $L_X \mathrm{g} = 0$ or $\nabla_{(i}X_{j)} = 0$,
2. $\Delta_c(X + \frac{n-2}{2n}\nabla_i X^i) = (X + \frac{n+2}{2n}\nabla_i X^i)\Delta_c$ only if X is conformal Killing vector field, i.e. $L_X \mathrm{g} = 2f\mathrm{g}$ or $\nabla_{(i}X_{j)} = f\mathrm{g}_{ij}$ for some $f \in \mathcal{C}^\infty(M)$.

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If $c = \frac{n-2}{4(n-1)}$ then $\Delta_c := \Delta_Y$ is the Yamabe Lapalcian. We get

$$\Delta_Y L_X^\lambda = \Delta_Y L_X^\mu, \qquad \text{for all } X \text{ CK vector fields,}$$

where $L_X^\lambda = \nabla_X + \lambda \nabla_i X^i$ is the Lie derivative along X on $\Gamma(|\Lambda^{\mathrm{top}} T^*M|^{\otimes\lambda})$ and $\lambda = \frac{n-2}{2n}$, $\mu = \frac{n+2}{2n}$.

Let $\mathcal{D}^{\lambda,\mu}(M) := \mathcal{D}(M; |\Lambda^{\mathrm{top}} T^*M|^{\otimes\lambda}, |\Lambda^{\mathrm{top}} T^*M|^{\otimes\mu})$. It is isomorphic to $\mathcal{D}(M)$ via

$$\begin{array}{ccc}
\Gamma(|\Lambda^{\mathrm{top}} T^*M|^{\otimes\lambda}) & \longrightarrow & \Gamma(|\Lambda^{\mathrm{top}} T^*M|^{\otimes\mu}) \\
\Big\uparrow {\scriptstyle |\mathrm{vol_g}|^{\lambda}} & & \Big\uparrow {\scriptstyle |\mathrm{vol_g}|^{\mu}} \\
\mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M)
\end{array}$$

Since $\mathrm{g} \mapsto \exp(2f)\mathrm{g}$ translates into $\mathrm{vol_g} \mapsto \exp(nf)\mathrm{vol_g}$, the usual transformation rule

$$\Delta_Y \mapsto \exp(-\frac{n+2}{2}f) \circ \Delta_Y \circ \exp(\frac{n-2}{2}f)$$

means that Δ_Y is conformally invariant in $\mathcal{D}^{\lambda,\mu}(M)$ if $\lambda = \frac{n-2}{2n}$ and $\mu = \frac{n+2}{2n}$.

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We restrict our study to $\Delta_Y \in \mathcal{D}^{\lambda,\mu}(M)$.

Symmetries of (null) geodesic flow on (M, g)

Let $\sigma_k : \mathcal{D}_k(M) \to \mathrm{Pol}_k(T^*M) \cong \Gamma(\mathcal{S}^k TM)$ be the principal symbol map at order k. We have $\sigma_2(\Delta_Y) := H$ where $H = \mathrm{g}^{ij} p_i p_j$ and (x^i, p_i) are coordinates on T^*M.

1. $[\Delta_Y, D_1] = 0 \Rightarrow \{H, \sigma_k(D_1)\} = 0$, i.e. $\sigma_k(D_1) = K$ is a Killing tensor, Killing equation $\nabla_{(i_0} K_{i_1 \cdots i_k)} = 0$.
2. $\Delta_Y D_1 - D_2 \Delta_Y = 0 \Rightarrow \{H, \sigma_k(D_1)\} \in (H)$, i.e. $\sigma_k(D_1) = K$ is a conformal Killing tensor, i.e. $\nabla_{(i_0} K_{i_1 \cdots i_k)_0} = 0$.

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Questions: does there exists

- $\mathcal{Q} : \{(\text{conformal) Killing tensors}\} \longrightarrow \{(\text{conformal) symmetries of } \Delta_Y\}$?
- extra conditions on K for it to give rise to (conformal) symmetries of Δ_Y?

Known results on (conformal) symmetries of Δ_Y

1. Carter proves in '77: for g a Ricci-flat metric we have $[\Delta_Y, \nabla_i K^{ij} \nabla_j] = 0$ iff K is a Killing 2-tensor. Moreover, he get in general

$$[\Delta, \nabla_i K^{ij} \nabla_j] = -\frac{2}{3}(\nabla_i K^{k[i} R_k^{j]}) \nabla_j .$$

2. Eastwood proves in '05: for g a conformally flat metric and K a conformal Killing tensor there exists (D_K, D'_K) such that $\Delta_Y D_K = D'_K \Delta_Y$. For K a 2-tensor we get

$$\begin{aligned} D_K &= K^{ij}\nabla_i\nabla_j + \frac{n}{n+2}(\nabla_i K^{ij})\nabla_j + \frac{n(n-2)}{4(n+2)(n+1)}(\nabla_i\nabla_j K^{ij}) - \frac{n+2}{4(n+1)} R_{ij}K^{ij}, \\ D'_K &= K^{ij}\nabla_i\nabla_j + \frac{n+4}{n+2}(\nabla_i K^{ij})\nabla_j + \frac{n+4}{4(n+1)}(\nabla_i\nabla_j K^{ij}) - \frac{n+2}{4(n+1)} R_{ij}K^{ij}. \end{aligned}$$

Conformally invariant quantization

Our startegy relies on a quantization map, i.e. a linear map $\mathcal{Q} : \mathrm{Pol}(T^*M) \to \mathcal{D}(M)$ such that $\sigma_k \circ \mathcal{Q} = \mathrm{Id}$ on degree k symbols.

Theorem (Mathonet, Radoux; Silhan '09)

For generic weights λ, μ, there exists a family of quantizations indexed by metrics

$$\mathcal{Q}_g^{\lambda,\mu} : \mathrm{Pol}^{\mu-\lambda}(T^*M) \to \mathcal{D}^{\lambda,\mu}(M),$$

which are natural, i.e.

$$\mathcal{Q}_{\Psi^* g}^{\lambda,\mu}(\Psi^* P) = \Psi^* \mathcal{Q}_g^{\lambda,\mu}(P),$$

for all symbol P, metric g and $\Psi \in \mathrm{Diff}(M)$, and conformally invariant i.e.

$$\mathcal{Q}_{\exp(2f)g}^{\lambda,\mu} = \mathcal{Q}_g^{\lambda,\mu}.$$

$$\begin{aligned} \mathcal{Q}_{\mathrm{g}}^{\lambda,\mu}(K) &= K^{ij}\nabla_i\nabla_j K + \beta_1(\nabla_i K^i j)\nabla_j + \beta_2 \mathrm{g}^{ij}(\nabla_i \mathrm{Tr} K)\nabla_j \\ &+ \beta_3(\nabla_i\nabla_j K^{ij}) + \beta_4(\Delta \mathrm{Tr} K) + \beta_5 \mathrm{Ric}_{ij} K^{ij} + \beta_6 R \,\mathrm{Tr} K. \end{aligned}$$

Main result (I)

Theorem

For $\lambda = \frac{n-2}{2n}$, $\mu = \frac{n+2}{2n}$, *and* K *a conformal Killing tensor, we get*

$$\Delta_Y \mathcal{Q}_g^{\lambda,\lambda}(K) - \mathcal{Q}_g^{\mu,\mu}(K)\Delta_Y = \mathcal{Q}_g^{\lambda,\mu}(\mathrm{Obs}(K))$$

where Obs *is given by the natural and conformally invariant operator*

$$\begin{aligned} \mathrm{Obs} : \mathrm{Pol}_2^0(T^*M) &\rightarrow \mathrm{Pol}_1^{\mu-\lambda}(T^*M) \\ K &\mapsto C^{ijkl}\nabla_j K_{kl} - \frac{4}{3}A^{ijk}K_{jk} \end{aligned}$$

with C *the Weyl tensor and* A *the Cotton-York tensor.*

Remind that C is the traceless part of the Riemann tensor and $A^{ijk} = -\frac{1}{n-3}\nabla_l C^{ijkl}$.

Main result (II)

Theorem

The operator $D_1 \in \mathcal{D}_2^{\lambda,\lambda}(M)$ *is a conformal symmetry of* Δ_Y *iff:* $K = \sigma_2(D_1)$ *is a conformal Killing tensor,* $\mathrm{Obs}(K)^\flat = d(f_K)$ *for some* $f_K \in \mathcal{C}^\infty(M)$ *and*

$$D_1 = \mathcal{Q}_g^{\lambda,\lambda}(K) + f_K + L_X^\lambda + c.$$

for some conformal Killing vector X *and constant* c*.*

Since $\mathcal{Q}_{\mathrm{g}}^{\lambda,\lambda}(K) = \mathcal{Q}_{\mathrm{g}}^{\mu,\mu}(K)$ for K Killing we deduce

Corollary

The operator $D \in \mathcal{D}_2^{\lambda,\lambda}(M)$ *is a symmetry of* Δ_Y *iff:* $K = \sigma_2(D_1)$ *is a Killing tensor,* $\mathrm{Obs}(K)^\flat = d(f_K)$ *for some* $f_K \in \mathcal{C}^\infty(M)$ *and*

$$D = \mathcal{Q}_g^{\lambda,\lambda}(K) + f_K + X + c.$$

for some Killing vector X *and constant* c*.*

Idea of proof

The pairing between λ- and μ-densities for $\lambda+\mu=1$ implies:

- there is an adjoint operation on $\mathcal{D}^{\lambda,\mu}(M)$ if $\lambda+\mu=1$, $\mathcal{Q}_{\mathrm{g}}^{\lambda,\mu}(K)^* = (-1)^k \mathcal{Q}^{\lambda,\mu}(\overline{K})$
- $\mathcal{Q}_{\mathrm{g}}^{\lambda,\lambda}(K)^* = (-1)^k \mathcal{Q}_{\mathrm{g}}^{\mu,\mu}(\overline{K})$.

Let $D_1 \in \mathcal{D}^{\lambda,\lambda}(M)$ and $Com(D_1) = \Delta_Y \mathcal{Q}_{\mathrm{g}}^{\lambda,\lambda}(K) - \mathcal{Q}_{\mathrm{g}}^{\mu,\mu}(K)\Delta_Y$ where $K = \left(\mathcal{Q}^{\lambda,\lambda}\right)^{-1}(D_1)$. Since $Com(D_1)$ is real and skew-adjoint we get

$$
\begin{array}{ccc}
\mathcal{D}_2^{\lambda,\lambda} & \xrightarrow{\quad Com \quad} & \mathcal{D}_3^{\lambda,\mu} \\
\uparrow {\scriptstyle \mathcal{Q}_{\mathrm{g}}^{\lambda,\lambda}} & & \uparrow {\scriptstyle \mathcal{Q}_{\mathrm{g}}^{\lambda,\mu}} \\
\mathrm{Pol}_2^0 & \xrightarrow[G_0 \oplus \mathrm{Obs}]{} & \mathrm{Pol}_3^{\mu-\lambda} \oplus \mathrm{Pol}_1^{\mu-\lambda}
\end{array}
$$

Naturality and conformal invariance determine uniquely the operators G_0 and Obs.

Examples of symmetries

- Obviously $\mathrm{Obs}(K)$ vanishes on conformally flat space and we recover Eastwood result.
- In the Ricci-flat case, $\mathrm{Obs}(K) = \frac{n-2}{8(n+1)} d(\Delta \mathrm{Tr} K)$ if K is a Killing, and we recover the Carter's result.
- In dimension 3 the pairs of diagonal metrics and Killing tensors are classified [Di Pirro 1896],

$$\begin{aligned} H &= \frac{1}{2(\gamma(x_1,x_2)+c(x_3))}\left(a(x_1,x_2)p_1^2 + b(x_1,x_2)p_2^2 + p_3^2\right), \\ K &= \frac{1}{\gamma(x_1,x_2)+c(x_3)}\left(c(x_3)a(x_1,x_2)p_1^2 + c(x_3)b(x_1,x_2)p_2^2 - \gamma(x_1,x_2)p_3^2\right). \end{aligned}$$

Then, we get $\mathrm{Obs}(K) = -\frac{3}{4} d(\mathrm{Ric}_0^{ij} K_{ij})$ with Ric_0 the traceless part of the Ricci tensor of g.

An example of obstructions to conformal symmetries

The Euclidean Taub-NUT metric:

$$g = \left(1 + \frac{2m}{r}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + \frac{4m^2}{1 + \frac{2m}{r}}\left(d\psi + \cos\theta d\phi\right)^2,$$

is hyperkähler

$$J_i = 4m\,(d\psi + \cos\theta d\phi) \wedge dx_i - \left(1 + \frac{2m}{r}\right)\varepsilon_{ijk}\,dx^j \wedge dx^k,$$

and admits a Killing-Yano tensor:

$$Y = 2m^2\,(d\psi + \cos\theta d\phi) \wedge dr + r(r+m)(r+2m)\sin\theta d\theta \wedge d\phi.$$

The skew-symmetric tensor $*Y$ is a conformal Killing-Yano tensor and J_i are Killing-Yano tensors hence

$$K_i = p_\mu p_\nu \left(*Y_\lambda^{(\mu} J_i^{\nu)\lambda}\right)$$

are conformal Killing tensors. We get $\mathrm{Obs}(K_i)$ is not exact.