Higher symmetries of the system $\Delta \oplus \not{D}$

Jean-Philippe Michel
(Université de Liège)

joint work with Josef Šilhan (Masaryk University)

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Overview on higher symmetries

Let E and F be two vector bundles over the smooth manifold M.

Definition

Let $D \in \mathcal{D}(M; E, F)$ be a differential operator. A higher symmetry of D is a diff. op. $A \in \mathcal{D}(M, E)$ such that

$$D \circ A = B \circ D$$

for some diff. op. $B \in \mathcal{D}(M, F)$.

- Higher symmetries (HS) preserve the kernel of D.
- The space of HS is a subalgebra of $\mathcal{D}(M, E)$, i.e., an associative and non-commutative filtered algebra.

- On $\mathbb{R}^n$, the algebras of HS are classified for: the Laplacian Δ [Eastwwod, Ann. Math. 2005], its powers Δ^k [Gover-Šilhan, JMP 2012], the CR-subLaplacian [Vlasáková, 2012], the Dirac operator $\not{D}$ [Eastwood-Somberg-Souček], the Schrödinger operator [Bekaert-Meunier-Moroz, JHEP 2012], the superLaplacian [Coulembier-Somberg-Souček, IMRN 2013].
- They all read as $\mathfrak{U}(\mathfrak{g})/\mathcal{J}$, with $\mathfrak{g} \hookrightarrow \mathrm{Vect}(\mathbb{R}^n)$ acting by Lie derivative on $\Gamma(E)$.

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For the Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$, acting on $\mathcal{E} := \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$:

- second order symmetries allow to classify separating coordinates for the Laplace equation $\Delta f = 0$;
- $\mathfrak{g} = \mathfrak{o}(n+2, \mathbb{C})$ and $\mathcal{J}$ is the Joseph ideal, so that $\ker \Delta$ is the minimal representation of $\mathrm{O}(p+1, q+1)$;
- $\mathfrak{U}(\mathfrak{g})/\mathcal{J}$ is an algebra of symmetries in higher spin field theories.

Conjecture

Let E, F be two irreducible homogeneous vector bundles over $M = G/P$. Assume $D \in \mathcal{D}(M; E, F)$ is a G-invariant diff. op.

Conjecture

The algebra of HS of D is the quotient $\mathfrak{U}(\mathfrak{g})/\mathcal{J}$, with $\mathcal{J}$ the annihilator of $\ker D$.

What about HS of systems of invariant differential operators?

Problematic

Let S be the spinor bundle over the pseudo-Euclidean space $(\mathbb{R}^n, \mathrm{g})$.

Determine the algebra of HS of the system of differential operators

$$\begin{array}{ccc} \mathcal{E}[-\frac{n-2}{2}] \oplus \mathcal{S}[-\frac{n-1}{2}] & \rightarrow & \mathcal{S}[-\frac{n+1}{2}] \oplus \mathcal{E}[-\frac{n+2}{2}] \\ \begin{pmatrix} f \\ \phi \end{pmatrix} & \mapsto & \begin{pmatrix} \Delta f \\ \not{D}\phi \end{pmatrix} \end{array}$$

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The HS read as

$$\begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix} \begin{pmatrix} a & \alpha^- \\ \alpha^+ & A \end{pmatrix} = \begin{pmatrix} b & \beta^+ \\ \beta^- & B \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}$$

with new symmetries:

$$\Delta\alpha^- = \beta^+\not{D} \quad \text{on } \mathcal{S}[-\tfrac{n-1}{2}] \qquad \text{and} \qquad \not{D}\alpha^+ = \beta^-\Delta \quad \text{on } \mathcal{E}[-\tfrac{n-2}{2}]$$

Higher symmetries of Laplacian [Eastwwod, Ann. Math. 2005][M., AIF 2014]

Examples: Up to constants, first order symmetries are given by

$$\Delta(X + \tfrac{n-2}{2n}\partial_i X^i) = (X + \tfrac{n+2}{2n}\partial_i X^i)\Delta,$$

where X is conformal Killing vector field, i.e. $L_X g \in [g]$ or $\nabla_{(i}X_{j)_0} = 0$. The Lie algebra of such vector fields is $\mathfrak{g} = \mathfrak{o}(n+2, \mathbb{C})$.

Definition: a symmetric conformal Killing k-tensor K is a solution of the equation $\nabla_{(i_0} K_{i_1 \cdots i_k)_0} = 0$.

Theorem

For all $w \in \mathbb{R}$, there exists a unique $\mathfrak{g}$-equivariant linear isomorphism $\mathcal{Q}_w : \Gamma(\mathcal{S}TM) \to \mathcal{D}_w(\mathbb{R}^n)$, such that $\mathcal{Q}_w(K) = K^{i_1 \cdots i_k}\partial_{i_1} \ldots \partial_{i_k} + l.o.t.$.
The map $\mathcal{Q}_{-\frac{n-2}{2}}$ induces a bijection between symmetric conformal Killing tensors and HS of Laplacian.

Higher symmetries of Dirac operator [Eastwwod-Somberg-Souček][M., Silhan]

Let $\gamma : \wedge T^*M[1] \to \mathrm{End} S$, so that $\not{D} = \gamma^i \partial_i$.

Examples: Up to constants, first order symmetries are [Benn-Charlton, 1997]:

$$X - \frac{1}{2}\gamma(\boldsymbol{d}X^\flat) + \tfrac{n-1}{2n}(\partial_i X^i),$$

$$g^{ij}\gamma(\iota_{e_i}K)\nabla_{e_j} - \frac{\kappa}{\kappa+1}\gamma(\boldsymbol{d}K) + \frac{n-\kappa}{2(n+1-\kappa)}\gamma(\delta K)$$

where X is conformal Killing vector field and K a conformal Killing κ-form.

Definition: a conformal Killing (k,κ)-tensor $K \in \Gamma(\mathcal{S}^k TM \otimes \wedge^\kappa T^*M[1])$ is a solution of $\Pi(\nabla_{(i_0}K_{i_1\cdots i_k)[j_1\cdots j_\kappa]}) = 0$, with Π a specific projection.

Theorem

For all $w \in \mathbb{R}$, there exists a unique $\mathfrak{g}$-equivariant linear isomorphism $\mathcal{Q}_w : \Gamma(\mathcal{S}TM \otimes \wedge T^*M[1]) \to \mathcal{D}(\mathbb{R}^n, S[w])$, s.t. $\mathcal{Q}_w(K) = K^{(i_1\cdots i_k)}_{[j_1\cdots j_\kappa]}\gamma^{j_1}\cdots\gamma^{j_\kappa}\partial_{i_1}\cdots\partial_{i_k} + lot.$ The map $\mathcal{Q}_{-\frac{n-1}{2}}$ induces a bijection between conformal Killing tensors and HS of the Dirac operator.

HS of the system Laplace + Dirac operator

Let $\varepsilon \in S^* \otimes S^*$ be the invariant pairing on S.

Examples: if $\Lambda \in \mathcal{S}[\frac{1}{2}]$ is a twistor spinor, $\nabla_i \Lambda = -\frac{1}{n}\gamma_i(\not{D}\Lambda)$, the following are symmetries [Wess-Zumino, Nucl. Phys. B 1974]:

$$\Delta \alpha_\Lambda^- = \beta_\Lambda^+ \not{D}, \quad \begin{cases} \alpha_\Lambda^-(\phi) = \varepsilon(\Lambda, \phi), \\ \beta_\Lambda^+(\not{D}\phi) = \varepsilon(\Lambda, \not{D}^2\phi) + \frac{2}{n}\varepsilon(\not{D}\Lambda, \not{D}\phi), \end{cases} \quad \phi \in \mathcal{S}[-\tfrac{n-1}{2}];$$

$$\not{D} \alpha_\Lambda^+ = \beta_\Lambda^- \Delta, \quad \begin{cases} \alpha_\Lambda^+(f) = \gamma^i(\Lambda)\partial_i f + \frac{n-2}{n}(\not{D}\Lambda)\cdot f, \\ \beta_\Lambda^-(\Delta f) = \Lambda \cdot \Delta f, \end{cases} \quad f \in \mathcal{E}[-\tfrac{n-2}{2}].$$

Theorem

The matrix of operators $\begin{pmatrix} a & \alpha^- \\ \alpha^+ & A \end{pmatrix}$ *is a HS iff*

- *a is a HS of Δ and A is a HS of $\not{D}$,*
- *$\alpha^- = \sum_i a_i \circ \alpha_{\Lambda_i}^-$, with a_i HS of Δ and $\alpha_{\Lambda_i}^-$ as above,*
- *$\alpha^+ = \sum_i \alpha_{\Lambda_i}^+ \circ a_i$, with a_i HS of Δ and $\alpha_{\Lambda_i}^+$ as above.*

Composition of twistor spinors actions

Lie (super-)algebra ?

Candidate: Conf. Killing vector fields $\oplus$ Twistor-spinors.

Hint from Rep. Theory:

- in odd dimension, $\mathrm{TwSp}\otimes\mathrm{TwSp}\cong\wedge^+\mathbb{C}^{n+2}\cong$ space of conf. Killing odd forms;
- in even dimension, $\mathrm{TwSp}\otimes\mathrm{TwSp}\cong\wedge\mathbb{C}^{n+2}\cong$ space of all conf. Killing forms, whereas $\mathrm{TwSp}^+\otimes\mathrm{TwSp}^-\cong\wedge^+\mathbb{C}^{n+2}\cong$ space of conf. Killing odd forms;

Fact: the composition $\alpha^+_{\Lambda'}\circ\alpha^-_{\Lambda}$ gives indeed rise to all HS of 1st order of $\not{D}$.

$\Rightarrow$ Algebra of HS is not generated by a Lie (super-)algebra!

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Idea: If dimension=3 (or 4 and $\not{D} : S^+ \to S^-$), then the 1st order symmetries of $\not{D}$ are all given by the Lie algebra $\mathfrak{o}(n+2, \mathbb{C}) \oplus \mathbb{C}$ and we have

$$\mathfrak{o}(5) \cong \mathfrak{sp}(4) \qquad \text{and} \qquad \mathfrak{o}(6) \cong \mathfrak{sl}(4).$$

Supergeometric reformulation

$\Pi S^* \cong \mathbb{R}^{3|2}$ is a supermanifold with sheaf of functions

$$\mathcal{O}(\Pi S^*) = \mathcal{E} \oplus \mathcal{S} \oplus \wedge^2 \mathcal{S}.$$

The pairing ε is a distinguished element of $\wedge^2 \mathcal{S} \cong \mathcal{E}$. We define

$$\boxslash : \mathcal{O}\Big(\Pi S^*[-\tfrac{1}{2}]\Big)[-\tfrac{n-2}{2}] \to \mathcal{O}\Big(\Pi S^*[-\tfrac{1}{2}]\Big)[-\tfrac{n}{2}]$$

by the formula $\boxslash := \varepsilon\Delta + \not{D} + \varepsilon^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \not{D} & 0 \\ \Delta & 0 & 0 \end{pmatrix}$.

Proposition

The algebra of HS of $\boxslash$ *is isomorphic to the one of* $\begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}$.

Twistor-spinors as odd vector fields

Let (x^i, θ^a) be coordinates on $\mathbb{R}^{3|2}$ and $(\partial_i, \partial_{\theta^a})$ the corresponding derivatives.

We define on $\mathcal{O}\left(\Pi S^*[-\frac{1}{2}]\right)[w]$

- $L_X = X^i\partial_i - \frac{1}{2}\gamma(\boldsymbol{d}X^\flat)_a^b\theta^a\partial_{\theta^b} - (\frac{w}{n} - \frac{1}{2n}\theta^a\partial_{\theta^a})(\partial_i X^i)$,
- $L_\Lambda^+ = \gamma^i{}_b^a\Lambda_a\theta^b\partial_i - 2(\frac{w}{n} + \frac{1}{n}\theta^a\partial_{\theta^a})(\not{D}\Lambda)$,
- $L_\Lambda^- = \varepsilon^{ab}\Lambda_a\partial_{\theta^b}$.

We have

$$\boxslash L_X = L_X\boxslash, \qquad \boxslash L_\Lambda^+ = L_\Lambda^-\boxslash, \qquad \boxslash L_\Lambda^- = L_\Lambda^+\boxslash\,.$$

Proposition

The space $\langle c, L_X\rangle \oplus \langle L_\Lambda^+, L_\Lambda^-\rangle$ is stable under the commutator in $\mathcal{D}_w(\mathbb{R}^{3|2})$ and isomorphic to the Lie superalgebra $\mathrm{spo}(4|2)$.

Main result

Theorem

The algebra of HS of of $\boxslash$ *and* $\begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}$ *is isomorphic to* $\mathfrak{U}(\mathfrak{spo}(4|2))/\mathcal{J}$*, with* $\mathcal{J}$ *the Joseph-like ideal.*

We have

$$\mathfrak{g} \odot \mathfrak{g} = \square\!\square\!\square\!\square \oplus \begin{smallmatrix}\square\!\square\\ \square\!\square\end{smallmatrix}_{0} \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}_{0} \oplus \mathbb{C}\cdot(\text{Casimir}).$$

$\mathcal{J}$ is generated by

$$\begin{smallmatrix}\square\!\square\\ \square\!\square\end{smallmatrix}_{0} \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}_{0} \oplus \mathbb{C}\cdot(\text{Casimir} - \rho).$$

Conclusion

HS of system of differential operators:

- new symmetries,
- new realization of representations,
- applications to higher spin field theories.