Conformally equivariant quantization of supercotangent bundles

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Abstract

Let (M,g) be a conformally flat spin manifold. Conformally equivariant quantization [DLO99, DO01], $\mathcal{Q}^{\lambda,\mu}:\mathcal{S}^\delta(M)\rightarrow\mathcal{D}^{\lambda,\mu}(M)$, is a conformally equivariant mapping between weighted symbols and differential operators acting on densities over M. We extend here this construction to super symbols, which are functions on the supercotangent bundle $\mathcal{M}$ of M, and to the space of differential operators acting on spinor densities over M. We obtain an explicit covariant expression of $\mathcal{Q}^{\lambda,\mu}$ on symbols of degree one in the even momenta, and prove existence and uniqueness of $\mathcal{Q}^{\lambda,\mu}$ in the generic case. Applications to conformally invariant differential operators and Killing-Yano tensors are given.

Equivariant quantization of cotangent bundles

Let G be a Lie group of Lie algebra $\mathfrak{g}$, and M be a manifold endowed with a G-flat structure. Then $\mathfrak{g}\subset\mathrm{Vect}(M)$. We denote by $\mathcal{F}^\lambda=\Gamma(|\Lambda^nT^*M|^{\otimes\lambda})$ the space of λ-densities.

1. *Phase space*: $\mathfrak{g}\circlearrowleft(T^*M,\omega_{T^*M})$, where $\omega_{T^*M}=dp_i\wedge dx^i$.
2. *Space of classical observables*: the $\mathrm{Vect}(M)$-module $\mathcal{S}^\delta(M)=\Gamma(\mathcal{S}TM)\otimes\mathcal{F}^\delta\subset\mathcal{C}^\infty(T^*M)\otimes\mathcal{F}^\delta$ of δ-weighted symbols, with $\delta=\mu-\lambda$.
3. *Space of quantum observables*: the $\mathrm{Vect}(M)$-module $\mathcal{D}^{\lambda,\mu}(M)$ of differential operators Aon densities, $A:\mathcal{F}^\lambda\rightarrow\mathcal{F}^\mu$.

Definition 0.1. *A G-equivariant quantization of the cotangent bundle T^*M is a morphism of $\mathfrak{g}$-modules, $\mathcal{Q}^{\lambda,\mu}:\mathcal{S}^\delta(M)\rightarrow\mathcal{D}^{\lambda,\mu}(M)$, preserving the principal symbol.*

- Existence and uniqueness of the conformally equivariant quantization [DLO99]. Explicit formula for $\mathcal{Q}^{\lambda,\mu}$ acting on symbols of degrees less than 2 [DO01].
- For $\lambda=\mu=\frac{1}{2}$, it defines a conformally invariant star-product [DEGO04].

Classical and quantum spaces of spin systems

Let (M,g) be a pseudo-riemannian spin manifold, representing the configuration space of a spin system.

1. *Phase space*: the supercotangent bundle $\mathcal{M}=T^*M\times_M\Pi TM$, with Π the parity reversing functor. We denote by (x^i,p_i,ξ^i) a system of natural local coordinates, generating locally $\mathcal{C}^\infty(\mathcal{M})=\mathcal{C}^\infty(T^*M)\otimes\Omega(M)$. Its canonical symplectic form is [Rot91]

$$\omega=dp_i\wedge dx^i+\frac{\hbar}{4\mathrm{i}}g_{lm}R^m_{kij}\xi^k\xi^l dx^i\wedge dx^j+\frac{\hbar}{2\mathrm{i}}g_{ij}d^\nabla\xi^i\wedge d^\nabla\xi^j. \tag{1}$$

2. *Space of classical observables*: the space of super δ-weighted symbols $\mathcal{S}^\delta[\xi]=\Gamma(\mathcal{S}TM\otimes\Lambda T^*M)\otimes\mathcal{F}^\delta$, included in $\mathcal{C}^\infty(\mathcal{M})\otimes\mathcal{F}^\delta$. It is graded by the degrees in p and in ξ: $\mathcal{S}^\delta(M)=\bigoplus_{k,\kappa}\mathcal{S}^\delta_{k,\kappa}[\xi]$.

3. *Space of quantum observables*: the space of differential operators acting on spinor densities $\mathsf{D}^{\lambda,\mu}$, filtered by the order: $\mathsf{D}^{\lambda,\mu}_0\subset\mathsf{D}^{\lambda,\mu}_1\subset\cdots$.

Let us introduce the Darboux coordinates given by the map ev_g on $\mathcal{M}$:

$$\mathsf{ev}_g(x^i)=x^i,\quad \tilde\xi^a=\mathsf{ev}_g(\xi^i)=\theta^a_i\xi^i\quad\text{and}\quad\tilde p_i=\mathsf{ev}_g(p_i)=p_i+\omega^a_{bi}\tilde\xi_a\tilde\xi^b,$$

where ω is the Levi-Civita connection and θ an orthonormal frame. On a chart U, the normal ordering is defined by [Get83]

$$\mathcal{N}:\mathcal{S}^\delta(U)[\xi]\rightarrow\mathsf{D}^{\lambda,\mu}(U)$$
$$P^{i_1\cdots i_k}_{j_1\cdots j_\kappa}(x)\tilde\xi^{j_1}\cdots\tilde\xi^{j_\kappa}\tilde p_{i_1}\cdots\tilde p_{i_k}\mapsto P^{i_1\cdots i_k}_{j_1\cdots j_\kappa}(x)\frac{\gamma^{j_1}}{\sqrt2}\cdots\frac{\gamma^{j_\kappa}}{\sqrt2}\frac{\hbar}{\mathrm{i}}\partial_{i_1}\cdots\frac{\hbar}{\mathrm{i}}\partial_{i_k}.$$

The geometric quantization Q_G of $(\mathcal{M},\omega)$

Definition 0.2. *[NT02] A N-structure on (M,g) of even dimension, at least 4, is a complex subbundle P of $T^{\mathbb{C}}M$ whose fibers are isotropic and maximal for g.*

Suppose that P defines a N-structure on (M,g). Then, $(\mathcal{M},\omega)$ admits a polarization, and polarized functions are sections of ΛP^*.

Theorem 0.3. • *$Q_G(\mathcal{S}_0[\xi])\simeq\mathrm{Cl}(M,g)$ turns ΛP^* in a spinor bundle of M.*

- *$Q_G(\mathcal{J}^{T^*M}_X)=\nabla_X$, where $\mathcal{J}^{T^*M}:T^*M\rightarrow(\mathrm{Vect}(M))^*$ is the momentum map and ∇_X is the spinor covariant derivative.*
- *Q_G is defined on $\mathcal{S}_1\oplus\mathcal{S}_0[\xi]$, and coincide with the normal ordering $\mathcal{N}$.*

Classical and quantum actions of the conformal Lie algebra $\mathrm{conf}(M,g)$

$$\mathrm{conf}(M,g)=\{X\in\mathrm{Vect}(M)\,|\,L_Xg=Fg,\ \text{for}\ 0<F\in\mathcal{C}^\infty(M)\}\le\mathrm{o}(p+1,q+1).$$

Let $\alpha=p_idx^i+\frac{\hbar}{2\mathrm{i}}g_{ij}\xi^id^\nabla\xi^j$ be the potential 1-form of $\mathcal{M}$, i.e. $d\alpha=\omega$, and let $\beta=g_{ij}\xi^idx^j$.

Theorem 0.4. *There is a unique lift $\mathrm{conf}(M,g)\ni X\mapsto\tilde X\in\mathrm{Vect}(\mathcal{M})$ preserving α and the direction of β.*

We denote by $\mathbb{L}^\delta_X=X^i\partial_i-p_j(\partial_iX^j)\partial_{p_i}+\xi^i(\partial_iX^j)\partial_{\xi^j}+\delta(\partial_iX^i)$ the natural action of X on $\mathcal{S}^\delta[\xi]$. Its hamiltonian action is given by $L^\delta_X=\tilde X+\delta\partial_iX^i$, i.e.,

$$L^\delta_X=\mathsf{ev}_g\mathbb{L}^\delta_X(\mathsf{ev}_g)^{-1}-\frac{\partial_iX^i}{n}\tilde\xi^i\partial_{\tilde\xi^i}-\frac{\hbar}{2\mathrm{i}}\tilde\xi_k\tilde\xi^j(\partial_i\partial_jX^k)\partial_{\tilde p_i}.$$

The Lie derivative of spinors [Kos72] (of weight λ) along $X\in\mathrm{conf}(M,g)$ is given by

$$\mathsf{L}^\lambda_X=Q(\mathcal{J}^{\mathcal{M}}_X)+\lambda\nabla_iX^i=\nabla_X+\frac{1}{4}\nabla_{[j}X_{i]}\gamma^i\gamma^j+\lambda\nabla_iX^i,$$

where $\mathcal{J}^{\mathcal{M}}:\mathcal{M}\rightarrow(\mathrm{conf}(M,g))^*$ is the momentum map. The quantum action of $X\in\mathrm{conf}(M,g)$ on $A\in\mathsf{D}^{\lambda,\mu}$ is defined as $\mathcal{L}^{\lambda,\mu}_XA=\mathsf{L}^\mu_XA-A\mathsf{L}^\lambda_X$.

Proposition 0.5. *$\mathcal{N}^{-1}\circ\mathcal{L}^{\lambda,\mu}_X\circ\mathcal{N}-L^\delta_X=$ nilpotent operator lowering the degree in p.*

Main results

Let (M,g) be a conformally flat spin manifold, of dimension n.

Theorem 0.6. *For δ generic, there exists a unique conformally equivariant quantization $\mathcal{Q}^{\lambda,\mu}:\mathcal{S}^\delta[\xi]\rightarrow\mathsf{D}^{\lambda,\mu}$.*

Theorem 0.7. *Let $\mathcal{Q}^{\lambda,\mu}:\mathcal{S}^\delta_1[\xi]\rightarrow\mathsf{D}^{\lambda,\mu}_1$ be a conformally equivariant quantization.*

1. *If $n\delta\notin\{1,\ldots,n+1\}$, then $\mathcal{Q}^{\lambda,\mu}$ exists and is unique.*
2. *If $n\delta=1$, or $n\delta=n+1$, and $\left(\lambda=\frac{n-1}{2n},\ \mu=\frac{n+1}{2n}\right)$ or $\left(\lambda=\frac{-1}{2n},\ \mu=\frac{2n+1}{2n}\right)$, then $\mathcal{Q}^{\lambda,\mu}$ exists but is not unique. With the additionnal condition $\mathcal{Q}^{\lambda,\mu}(\bar P)=\mathcal{Q}^{\lambda,\mu}(P)^*$, uniqueness is recovered. These values are called resonances.*
3. *Else, there exists no such $\mathcal{Q}^{\lambda,\mu}$.*

Let $\mathcal{T}^\delta$ be the $\mathrm{conf}(M,g)$-module $\left(\bigoplus_\kappa\mathcal{S}^{\delta-\frac{\kappa}{n}}_{*,\kappa},\mathbb{L}^\delta_X\right)$. The conformal equivariant superization $\mathrm{S}^\delta_{\mathcal{T}}:\mathcal{T}^\delta\rightarrow\mathcal{S}^\delta[\xi]$ exists and is unique in the generic case. Restricting $\mathrm{S}^\delta_{\mathcal{T}}$ to symbols of degree 1 in p, the critical values of δ are $\delta=\frac{2}{n},\cdots,\frac{n}{n}$.

In terms of ∂^∇ the horizontal derivation, $\mathcal{Q}^{\lambda,\mu}$ has for expression in the generic case,

$$\boxed{\begin{aligned}\mathcal{Q}^{\lambda,\mu}(P)=&\ \mathcal{N}(P^i)\tfrac{\hbar}{\mathrm{i}}\nabla^\lambda_i\\&+\mathcal{N}\left((c_d+c_{\lambda\psi})\,\partial^\nabla_iP^i-c_{\lambda\psi}\,g^{ij}g_{kl}\xi^l\partial^\nabla_iP^k_j+c_{\gamma\omega}\,\xi^i\partial^\nabla_iP^j_j+c_{\lambda\omega}\,g^{ij}\partial^\nabla_iP^k_{kj}\right),\end{aligned}} \tag{2}$$

where $P^i=\partial_{p_i}P$, $P_i=\partial_{\xi^i}P$, and the coefficients are functions of the degree of P in ξ.

Applications and examples

All the following results stand on (M,g) a conformally flat spin manifold, of dimension n.

Proposition 0.8. $\mathcal{J}^{T^*M}_X\xrightarrow{\mathrm{S}^0_{\mathcal{T}}}\mathcal{J}^{\mathcal{M}}_X\xrightarrow{\mathcal{Q}^{\lambda,\lambda}}\mathsf{L}^\lambda$.

Let $\Delta=\tilde p_i\tilde\xi^i$, $\chi=(\mathrm{vol}_g)_{i_1\cdots i_n}\tilde\xi^{i_1}\cdots\tilde\xi^{i_n}$ and $\Delta*\chi$ be their Moyal star-product w.r.t ω.

Theorem 0.9. *The conformally invariant elements of modules $\mathcal{T}$, $\mathcal{S}$ and D are*

- *$\mathsf{ev}_g^{-1}\left(\Delta^a*\chi^bR^s\right)\in\mathcal{T}^{\frac{2s+a}{n}}$, where $a,b=0,1$ and $s\in\mathbb{N}$;*
- *$\Delta^a*\chi^bR^s\in\mathcal{S}^{\frac{2s+a}{n}}[\xi]$, where $s\in\mathbb{N}$ and $a,b=0,1$ s.t. $a+b\neq0$;*
- *$\mathcal{N}(\chi)\in\mathsf{D}^{\lambda,\lambda}$, or $\mathcal{N}(\Delta*\chi)\in\mathsf{D}^{\frac{n-1}{2n},\frac{n+1}{2n}}$, or $\mathcal{N}(\Delta R^s)\in\mathsf{D}^{\frac{n-2s-1}{2n},\frac{n+2s+1}{2n}}$, for all $\lambda\in\mathbb{R}$ and $s\in\mathbb{N}$.*

When $\mathrm{S}^\delta_{\mathcal{T}}$ and $\mathcal{Q}^{\lambda,\mu}$ exist, we have: $(\mathcal{T}^\delta)^{\mathrm{conf}}\xrightarrow{\mathrm{S}^\delta_{\mathcal{T}}}(\mathcal{S}^\delta)^{\mathrm{conf}}\xrightarrow{\mathcal{Q}^{\lambda,\mu}}(\mathsf{D}^{\lambda,\mu})^{\mathrm{conf}}$.

Corollary 0.10. *The Dirac operator arises as the conformally equivariant quantization of the conformal invariant symbol $\Delta\in\mathcal{S}^{\frac{1}{n}}$,*

$$\boxed{\mathcal{Q}^{\frac{n-1}{2n},\frac{n+1}{2n}}(\Delta)=\frac{\gamma^i}{\sqrt2}\nabla_i\in\mathcal{D}^{\frac{n-1}{2n},\frac{n+1}{2n}},\quad(3)}$$

the weights $\left(\frac{n-1}{2n},\frac{n+1}{2n}\right)$ corresponding to one of the two resonances of $\mathcal{Q}^{\lambda,\mu}$.

Definition 0.11. *A conformal Killing-Yano (CKY) tensor of order κ is a κ-form f satisfying*

$$\nabla_{(j_1}f_{j_2)j_3\ldots j_{\kappa+1}}=g_{j_1j_2}\Phi_{j_3\ldots j_{\kappa+1}}-(\kappa-1)g_{[j_3(j_1}\Phi_{j_2)j_4\ldots j_{\kappa+1}]}, \tag{4}$$

where $\Phi=\frac{1}{n-\kappa+1}\nabla_{j_1}f^{j_1}_{j_2\ldots j_\kappa}$. If $\Phi=0$, f is a Killing-Yano tensor.

For f a tensor of order κ, we denote by $P_f=f^i_{j_1\ldots j_{\kappa-1}}\xi^{j_1}\ldots\xi^{j_{\kappa-1}}p_i\in\mathcal{T}^0$.

Theorem 0.12 (Generalization of [Tan95]).

$$\boxed{\{\Delta,\mathcal{S}^0_{\mathcal{T}}(P_f)\}=0\ (\propto\Delta)\iff f\ \text{is a (conformal) Killing-Yano tensor.}} \tag{5}$$

Proposition 0.13. *Let f be a Killing-Yano tensor of order 2, then*

$$\mathcal{Q}^{\lambda,\lambda}(\mathrm{S}^0_{\mathcal{T}}(P_f))=\frac{\hbar}{\mathrm{i}\sqrt2}\left(f^i_j\gamma^j\nabla_i+\frac{1}{6}\gamma^i\gamma^j\gamma^k\nabla_if_{jk}\right). \tag{6}$$

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